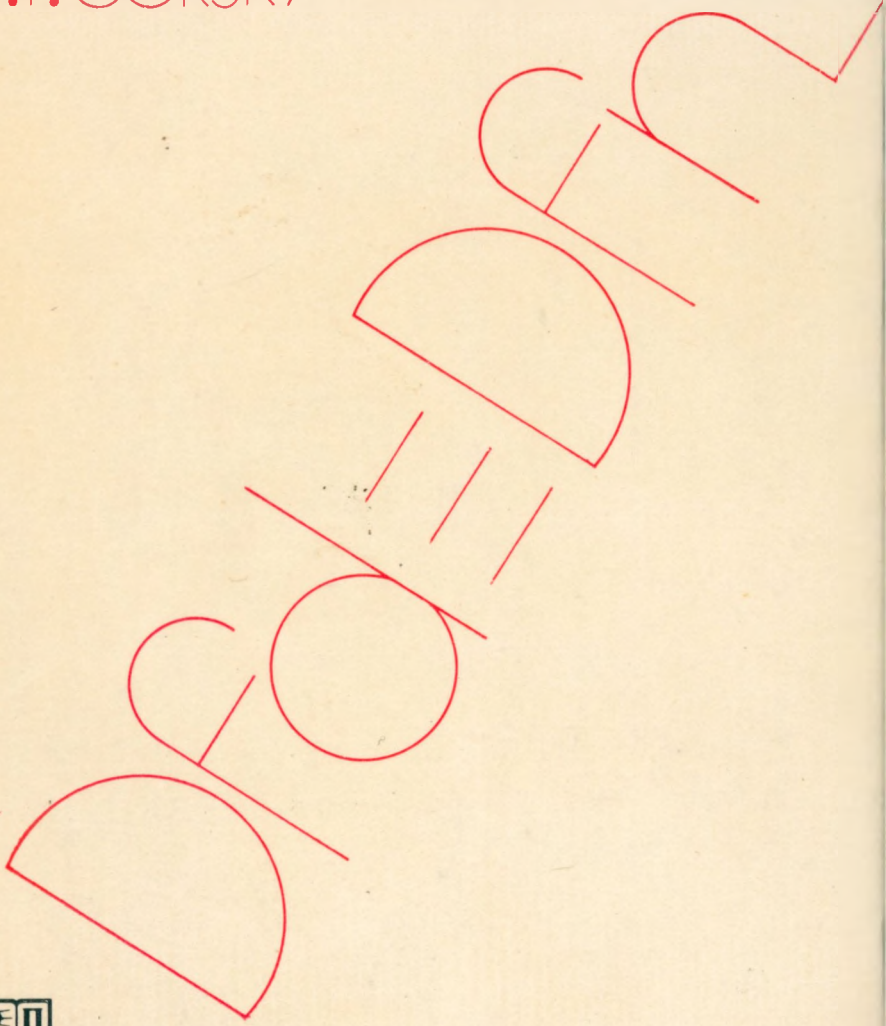
Definition

D. P. GORSKY



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Books on definition are few and far between in the logical literature. The problem itself, however, has been a matter of acute debate both with specialists on logic and with natural scientists.

This book by Professor D. P. Gorsky, Ph. D., considers a wide range of logical and methodological problems involved in the theory of definition. The author analyses numerous types and kinds of definition and their specific features and elucidates the essence of definition of concepts as well as rules for their application.

D. P. GORSKY

Definition

(Logico-Methodological Problems)

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Д. П. Горский

ОПРЕДЕЛЕНИЕ (Логико-методологические проблемы)

На английском языке

REQUEST TO READERS

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This book is dedicated to the memory of my mother, Varvara Vassilyevna Gorskaya

FOREWORD

The deductive method of reasoning developed by mathematical (or symbolic) logic is not commonly applied in an *explicit* form either in everyday reasoning or even in such a rigorous science as mathematics. This does not mean, however, that deduction theory (metamathematics included) is of purely theoretical interest. It is widely used in works on the foundations of mathematics, in solving some mathematical problems, and in technology. The ideas and methods of symbolic logic also came to be widely used in the analysis of certain fragments and components of scientific knowledge in general (the logic of scientific cognition).

Logic contains a branch known as the theory of definition which studies components that are explicitly expressed or implied in any more or less complete discourse, both everyday and scientific, in any conception or theory. We know, for instance, that definitions are used in all mathematical disciplines and in all mathematical reasoning to point out the characteristics of the objects under analysis. In proving theorems, a mathematician may not use any characteristics of objects other than those formulated in the definitions (explicit or implicit) or those that may be logically deduced from them.

In formulating definitions in the most

diverse theories and types of reasoning, the questions frequently arise whether the definitions of terms are correct, or why a certain definition should be accepted over another, or why a given definition requires generalisation, on what grounds a certain correct definition should be regarded as more effective than another correct definition, why different definitions are judged by different criteria of rigorousness, etc. A discussion of these and similar logical and methodological problems belongs in the theory of definitions and is the subject-matter of this book.

A logico-semantic and methodological analysis of definitions is applied in solving certain complicated and fundamental problems of scientific cognition; suffice it to recall that many debatable problems of science were solved through a more precise formulation of existing definitions, through demonstrating the equivalence of the definitions under analysis.

Chapter 2 treats the problem of definition in general and definition rules. The reason for this is that the concept of definition cannot be made more precise or substantiated in a proper way unless definition rules are considered. The analysis of the latter, however, calls for a discussion of the differences between analytic and synthetic, explicit and implicit, nominal and real, semantic and syntactical, complete and incomplete definitions, and other types of definition. In other words, it requires a preliminary introduction into the classification and description of the various types of definition used in natural language and in various sciences.

Although special methodological problems are dealt with in Chapters 6-9, the reader will find many remarks on this subject in the preceding chapters as well.

CHAPTER 1

TYPES OF DEFINITION

1.1. NOMINAL AND REAL DEFINITIONS

A definition is usually interpreted as a logical device enabling one (a) to distinguish, find, or construct the object that interests one; (b) to make the meaning of a term already current in science more precise, and also to form the meaning of a newly introduced term. Since the knowledge of the distinctive features of objects and of the meanings of the terms in question entails the formation of concepts of the objects defined, logicians often speak of definitions of concepts in question rather than of definitions of objects and meanings of terms.

'Definition' covers both the process of elaborating a certain proposition and the result of this process, i.e., the proposition itself. That which is defined is called *definiendum* (abbr. *Dfd*), and that which serves as the means of defining something bears the name of *definiens* (abbr. *Dfn*).

The division of definitions into nominal and real depends, first and foremost, on what is defined—the meaning or sense of the term[1], or the object itself. The conceptualist tradition in logic, particularly strongly defended in the German philosophy and logic, beginning with Christian von Wolff and Immanuel Kant and ending with Heinrich Rickert, held the view that definition reveals the content of the concept. Hence the expression 'definition of the concept', which is a literal translation of the corresponding German expression. However, some philosophers and logicians of the conceptualist persuasion were inclined to interpret definitions as nominal ones: in defining a concept, we define, first and foremost, the meaning of the corresponding term. Other

[1] Unless otherwise stipulated, the expressions 'the meaning of the term' and the 'sense of the term' will be treated as synonymous.

conceptualistically minded philosophers and logicians were inclined to interpret them as real: in defining a concept, we define in the first place the extensional of the concept, the object reflected by the corresponding concept. The two respective viewpoints on what is defined are therefore most frequently expressed by posing the alternative: is a given definition nominal or real? In other words: do we define the term or the corresponding object?

The first European logician to study the logical procedure of defining the essence of things seems to have been Democritus. Aristotle points out that 'our predecessors ... were not in possession of the notion of essence, nor of any definition of substance. The first who came near it was Democritus, and he was far from adopting it as a necessary method in natural science, but was merely brought to it, in spite of himself, by constraint of facts.'[1] Plato evidently reduces all definitions to real ones. In his *Theactetus* he states that *logos* in Greek is used to denote a feature that may serve to distinguish the thing in question from all other things.[2] Aristotle views definition as real, first and foremost. In his *Topica* and *Analytica* he indicates that definition concerns essential nature.[3] He emphasises the fact that definition is speech and cannot consist of a single word. According to Aristotle, an individual material object cannot be defined. Only that general feature which exists in many individual objects is definable. That general feature ('the second essence') is given a specific name (e.g., the common name for the general property which individual men possess is the word 'man'). Then the essence of man may be defined like this: 'Man is an animal endowed with reason.' The property specifying the second essence cannot merely be a proper (distinguishing) feature: it must necessarily be an essential feature (a specific kind of a proper feature).

However, when Aristotle deals with definitions as components of mathematical theories in his discussion of the initial self-obvious principles of scientific theories (axioms, postulates, and basic concepts), he treats definition as definition of

[1] *The Works of Aristotle*. Vol. V, Oxford at the Clarendon Press, 1912, 'De Partibus Animalium', Book I, 1.

[2] See *The Dialogues of Plato* in 4 Vols., Vol. I ('Theactetus'), Oxford at the Clarendon Press, 1953, Paras. 206c-208c.

[3] See *The Works of Aristotle*. Vol. I, Oxford at the Clarendon Press, 1928, 'Analytica Posteriora', Book II, 3.

the meanings of words, i.e., he regards it as nominal. In this case the 'definition will be a statement of the meaning of the name, or of an equivalent nominal formula. A definition in this sense tells you, e.g., the meaning of the phrase "triangular character".'[1] The substantiation of undefined basic concepts introduced into a theory, as well as other principles of science, belong, generally speaking, within the scope of philosophy, not mathematics. In Aristotle's *Metaphysics* they are analysed in their essence and are defined on the basis of the categories (matter, form, object, quantity, magnitude, continuity, discreteness, etc.), that is, they are introduced on the basis of real definitions which lie, strictly speaking, beyond the scope of the theories in question.[2]

More recently most philosophers, particularly those opposed to Aristotle, came to stress the nominal aspect of definition. Thus according to Locke, definition is 'nothing but making another understand by words what idea the term defined stands for'.[3] Pascal in his *L'Esprit geométrique* (The Spirit of Geometry) stresses the significance of nominal definitions for mathematics: 'In geometry the only definitions which are accepted are those which logicians call nominal definitions, that is to say only impositions of names to things which have been clearly designated in terms which are perfectly well known.... Their utility and their use is to clarify and to abridge discourse.'[4]

In the 19th century, Wilhelm Wundt made an attempt to reduce all definitions to nominal ones. In 20th-century methodology and logic, despite the wide-spread positivist and nominalist conceptions in the bourgeois countries, most authors regard as justifiable the division of all definitions into nominal and real.[5]

Walter Dubislav, a 20th-century German logician and mathematician, correctly divides all theories of definition into the following classes (depending on what is defined):

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[1] *The Works of Aristotle*. Vol. I. Analytica Posteriora', Book II, 9.

[2] See S. A. Yanovskaya, 'From the History of Axiomatic Theories', *Istoriko-matematicheskiye issledovaniya* (Studies in the History of Mathematics), Gostekhizdat Publishers, No. XI; Moscow, 1958, pp. 72-73.

[3] J. Locke. *An Essay Concerning Human Understanding*. Routledge and Kegan Paul, London, p. 329.

[4] Quoted from H. Meyer. 'On Definitions in Symbolic Logic'. In: *Synthese*, Vol. V, Nos. 5 and 6, 1946, p. 261.

[5] J. M. Bochenski. *Die zeitgenossischen Denkmethoden*. Erancke, Bern-München, 1965, S. 90.

(1) those revealing the essence of the object defined (e.g., Aristotle, Hegel);

(2) those explicating the content of concepts (e.g., Christian von Wolff, Kant);

(3) those establishing or clarifying the meaning of a symbolic expression already current in the language (Thomas Hobbes, Blaise Pascal) or fixing the meanings of newly introduced symbolic expressions in the language of science.[1] This tradition was continued by philosophers and logicians working on problems of mathematical logic and the analysis of the language of science. In formulating general definitions of definition they therefore emphasised the nominal nature of all definition.

Thus, Bertrand Russell and Alfred Whitehead believe that 'a definition is a declaration that a certain newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols of which the meaning is already known'.[2] Ludwig Wittgenstein writes that 'definitions are rules for the translation of one language into another'.[3] Rudolf Carnap interprets definition as a rule for the mutual transformation of words in the same language[4] or as explication of terms.

H. S. Leonard in his book *An Introduction to Principles of Right Reason*[5] assumes that any definition deals with revealing or establishing the meaning of a term. Therefore his division of definitions into nominal and real concerns nominal definitions in the broad sense.

Leonard interprets real definitions as those establishing or explicitly describing the meanings of terms on the basis of their common usage. If a novel term is introduced that has not figured in science or language in general, the corresponding definition will also be real if it is recommended for wide use. By 'nominal definition' Leonard means a clarification of a term in context, which may vary depending on circum-

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[1] W. Dubislav. *Die Definition*. Meiner, Leipzig, 1931, S. 2-27.

[2] A. Whitehead, B. Russell. *Principia Mathematica*. Vol. I, Cambridge Univ. Press, Cambridge, 1925, p. 11.

[3] L. Wittgenstein. *Tractatus Logico-Philosophicus*. Routledge and Kegan Paul, London, 1949, p. 59.

[4] R. Carnap. 'Foundations of Logic and Mathematics'. In: *International Encyclopedia of Unified Science*, Vol. I, No. 3, Univ. of Chicago Press, Chicago, 1939, p. 39.

[5] H. S. Leonard. *An Introduction to Principles of Right Reason*. Henry Holt and Company, New York, 1957, pp. 275-77.

stances (changes in the meaning of the context, the attitudes of the subject formulating the definition, etc.).

Supposing that in talking to one's colleagues one clarifies the meaning of the expression 'a person prepared to take entrance exams in maths' with the words 'a person prepared to take entrance exams in maths will be taken to mean a person capable of solving a given set of problems in two hours'. In Leonard's view, we are dealing here with a nominal definition.

As we see it, Leonard's division of definition according to the criterion indicated above is in a sense justified. This division, however, can in no way replace the traditional division of definition into nominal and real based on quite a different criterion. To endow a term already used in science with quite a different meaning, and to do it with any sort of justification, one should prove that the division of definition into nominal and real now current in philosophy, logic, and methodology of science is unjustified. Leonard's book, however, contains no such proofs.

Richard Robinson in his book *Definition*[1] divides all definition into real and nominal. He is subdividing the latter into word-word and word-thing definitions. Word-word definitions have this form: a certain expression has the same meaning as some other expression. If, for instance, someone states that the French word *rouge*, unfamiliar to you, has the same meaning as the English word 'red', that will be a word-word definition of the word *rouge*. Word-thing definitions have this form: a given verbal expression is used to denote a certain object (or a certain class of objects). Word-thing definitions are divided into lexical and stipulative. Lexical definitions are definitions of unfamiliar expressions through descriptions of their meanings. Stipulative definitions are definitions of unfamiliar expressions which are used in various contexts in different meanings and which assume a procedure for choosing one of the meanings suitable for the given case.

Word-word definitions, which are translations of expressions from one language into another, will not be here considered at all.

What are, then, nominal definitions and real definitions?

A *nominal definition* is a definition which serves

[1] R. Robinson. *Definition*. Oxford at the Clarendon Press, 1950.

(a) to formulate in an explicit form the meaning of a term already current in the language of science or in a natural language (including symbols of the artificial languages of science).

A clarification or specification of the meaning of an already existing term is sometimes interpreted as its introduction into science or into some natural language at a different level of cognition (and consequently as an innovation: a new and more precise meaning for an already existing sign expression is introduced);

(b) to fix the meaning of a term newly introduced into a natural language or the language of science. In this case, a new meaning for a new sign expression is introduced;

(c) to record the fact that the terms of the definiendum (Dfd) and of the definiens (Dfn) denote identical objects;

(d) to introduce a new term as a mere abbreviation of another, usually more complicated, expression. Here we ignore temporarily the meanings of the Dfd and Dfn terms and consider them simply as sign forms (these definitions as distinct from other types of nominal definitions we shall call *nominal in the narrow sense of the word*).

A *real definition* is one that specifies and unambiguously distinguishes the object in question from other objects of a given domain. This specification will be effective if the extent of the classes *represented* by the terms Dfd and Dfn is the same (or, which is much the same thing, when the objects *represented* by the terms Dfd and Dfn are identical). Let us note that the word 'represented' in the formulations above has a terminological meaning which will be specified later.

In specifying the object in question through a real definition, we sometimes attempt simultaneously to characterise the Dfd in its essentials, to find the specific causes for the emergence or the modes of genesis of the object defined, and to reveal its structure.

The difference between nominal and real definitions in the characterisations given above remains indefinite at the level of logico-semantic analysis. It is therefore necessary to formulate this difference in a way that would introduce more effective criteria for recognising nominal and real definitions. To achieve this, let us introduce some distinctions in the characterisation of the use of terms, distinctions that

are current in the literature on mathematical logic and logical semantics.

Let us distinguish between the employment of a term in the functions of *mentioning* and *using*. The concept of employing a term in the function of mentioning will be somewhat generalised as compared with its interpretation by, let us say, Alonzo Church.[1] We do not restrict the employment of the term in the function of mentioning to its *autonymous* employment, that is, as a sign of itself; it will be extended also to cases in which the term's semantic characteristics will be discussed. We shall say that a term is *mentioned* in such cases (at least) when

(a) it is used autonymously, i.e., as a sign of itself;

(b) something is stated about it as true, but to establish truth or falsity of this statement one must have recourse to analysis of its extensional characteristics;

(c) one deals with the sphere of application of a term, indicating this sphere by reference to the corresponding set of objects given by its intensional or extensional characteristics; [2]

(d) the meaning of some sign expression is established through stating that it is synonymous with another sign expression;

(e) a new sign expression is introduced to abbreviate another (usually more cumbersome) sign expression.

In all these cases we shall deal with terms, their meanings being either disregarded or established in the definition; we shall deal with formal characteristics of terms (taking into account their meanings as well), and with replacement of one sign expression by another.

Let us illustrate the various employments of terms in the function of mentioning.

1. We deal with autonymous employment of a sign

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[1] Church believes that a term is employed in the function of mentioning only when it is autonymous (see A. Church. *Introduction to Mathematical Logic*. Vol. 1, Princeton University Press, Princeton, 1956, p. 64). In using a sign expression autonymously we ignore the signified, regarding the term as an independent material object, as a sign of itself. For instance, in the sentence 'The word "word" consists of four letters' the expression in double quotes is used autonymously.

[2] Extensional characteristics refer to the sphere of application of a sign; intensional characteristics refer to the content of the concept, to the structure of the sign expression.

expression when the latter is viewed as a certain material object regardless of its interpretation (meaning and sense). Such is the employment of the sign expression 'man' in the sentence ' "Man" consists of three letters'. To stress that we deal here with autonymous employment of a term, certain words are added (in spoken language), as in 'The word "man" consists of three letters', or the autonymous word is used in quotes or italicised (in written language), as in '"Man" consists of three letters', '*Man* consists of three letters'. It is advisable to disregard the interpretation of autonymously employed sign expression cited above when we are, for instance, analysing the structure or the formal properties of sign expressions.

2. Let us consider the sentence 'The word "Ruth" is a woman's name'. This sentence states something about the sign expression. The truth of this statement, however, cannot be substantiated by analysing the purely formal properties of this word. To substantiate its truth, we should have recourse to semantic analysis, pointing out that 'Ruth' is a proper name and is used in English to refer to females. This has to be done, since a word with the same formal syntactical features may also refer to males (cf. 'The word "Dick" is a man's name').

3. The following are instances of mentioning sign expressions which presuppose ascribing meaning to them through a description of their intensional or extensional characteristics: 'The word "pentagon" is used to denote a polygon with five sides', 'The phrase "domestic animals" is used to denote cows, horses, pigs, sheep, goats, donkeys, camels, cats, dogs, etc.'. In these cases sign expressions are characterised precisely from the point of view of their meanings through a description of their intensional or extensional characteristics.

4. One deals with employment of a sign expression in the function of mentioning when that sign expression presupposes the establishment of synonymity of two sign expressions in the following case: 'The word "mankind" has the same meaning in English as the word "humanity".'

5. The following is an example of employment of a sign expression in the function of mentioning, which assumes that one expression may be used as an abbreviation of another, more complex, expression: 'Instead of using the phrase "oscillations whose amplitude, together with energy, de-

creases with time" we shall use a simpler expression "decaying oscillations".'

We may use a shorter expression instead of a longer one provided we disregard entirely the interpretations of these sign expressions. One may stipulate, for example, that whenever we come across a phrasal expression (e.g., 'an equilateral rectangle'), we shall replace it by another and simpler expression consisting of fewer words (e.g., one word 'square').

In employing a certain term in the function of *using*, one is not dealing with a characterisation of the sign expression (considered together with its interpretation or independent of it), but with the object itself (material or abstract), its properties or correlations, the object being in a certain sense independent of the concrete modes of its language coding. In this case, terms function as representatives of objects or their indicators. As for the context, it deals with objects themselves: something is stated about them, certain correlations are established between them, something is asserted or negated about them, certain values are ascribed to them, etc. The check that we get at the cloakroom when handing over our overcoat to the cloakroom attendant performs a function similar to an application of the term: it indicates the object which it stands for and provides criteria for its retrieval; it carries information about the identity of an object belonging to a certain person with that very object. (Similarly, words and phrases carry information about the objects which they represent, thereby ensuring communication through speech.) When a person loses the check he is worried about the possibility of losing that which it stands for, rather than about the loss of the check as a particular individual object.[1]

Thus, in the sentences 'The snow is white', 'Give me a glass of water' the terms 'snow' and 'a glass of water' are

[1] One might apparently analyse, along these lines, the difference between using terms in the function of representation and that of signification, that is so clearly seen in the natural languages. In speech activity, the function of signifying an object by a sign is only manifested when a sentence states explicitly that a certain sign expression names or signifies certain objects (cf. sentences expressing semantic definitions, see 1.2). In most cases, however, sign expressions are employed in speech in their use function, thereby being representations of or substitutes for the objects in question. In studying speech activity at the linguistic and logical levels, we treat all relations between sign and object as the relation of signification or naming.

employed in the use function. What is characterised in the first sentence is the material object, the snow, and not the word 'snow': it is asserted that a certain object, and not the word, is white. In the second sentence the idea is to get a glass of water that can be drunk, and not the phrase 'a glass of water'. All the other words in the sentences cited are clearly employed in the use function. One may therefore define nominal and real definitions as follows:

A *nominal* definition will be taken to mean a definition in which that which is to be defined (Dfd) is a term in the function of *mentioning*.

A *real* definition will be taken to mean a definition in which the definiendum (Dfd) is an object (real, abstract, or imagined), since the term corresponding to the object in question is employed in the function of *using*.

The following two subclasses are to be singled out among the nominal definitions, first and foremost:

(a) nominal definitions in which the Dfd is a term used in the function of mentioning taken together with its meaning;

(b) nominal definitions in which the Dfd and the Dfn are sign expressions taken irrespective of their meanings. Here the Dfd is introduced as a contraction of the Dfn. We have termed them nominal definitions in the narrow sense.

Nominal definitions of the first subclass may be broken down into further two kinds: definitions in which the sign expression functioning as the Dfd is ascribed meaning through indicating a class of objects given either extensionally or intensionally (cf. 'The phrase "domestic animals" is used to denote cows, horses, pigs, etc.', 'The term "insulin" will be used to denote the secretion having certain stated properties'). The Dfd term is employed here in the function of mentioning, while the Dfn term in the use function. These definitions are called semantic definitions (see 1.2). In the second example, strictly speaking, the term 'insulin' introduced into the language is first used irrespective of its meaning, but it acquires the appropriate meaning in the context of the definition as a whole. In saying that in the nominal definitions considered here the Dfd is a term mentioned together with its meaning, we take into account the context of the definition in which it is introduced.

The second type of the subclass (a) nominal definitions

will encompass definitions in which both the Dfd and the Dfn terms are used in the function of mentioning (e.g., the expression 'decaying oscillations' means precisely the same as 'oscillations whose amplitude, together with energy, decreases with time').

It has been pointed out that in real definitions, as distinct from nominal ones, the Dfd and Dfn terms are simultaneously employed in the functions of using and representation. They are therefore usually interpreted as sentences in which the Dfd is a certain object, while the Dfn is the properties of that object enabling one to specify it. Answers to questions of the form 'What is an anticyclone?', 'What is an atom?', 'How is one to define the square?' are real definitions: they list the specific features of anticyclones, atoms, and squares enabling one to distinguish them from other objects.

The object defined may be specified not only through pointing out its distinctive features but also through listing operations performed upon it and the effects obtained, through a description of its interrelations with other objects of the domain, of its structure, genesis, etc.

Elaboration of nominal and real definitions is conditioned on sign activity involving formation of sentences of definite structure, analysis of meanings of their constituent terms, etc., as well as on activity involving the study of the corresponding objects and the use of the previous information about them. Sign activity does not play the same role in forming nominal and real definitions. In elaborating nominal definitions, sign activity is the starting point, the basis: we begin by introducing sign expressions, by analysing meanings, terms, etc., by discussing the problem of what term should be used to reduce a complex sign expression. In working out real definitions, we proceed, on the contrary, from the analysis of the objects in question (although they are *represented* in sign form), while the sign activity proper recedes into the background.

The Dfd in any definition may either be an object represented by a sign expression or the sign expression itself, viewed either in connection with its meaning or irrespective of it. In dealing with definitions of meanings of sign expressions in meaningful discourse and meaningful theories ('nominal definition'), we shall have to treat of the functions and properties of the sign expression itself. In these cases we

shall use the expression 'the Dfd term' (something that we have done already).

Real definitions may be used to define not only objects corresponding to names but also objects corresponding to predicators[1] (i.e., predicates), to sign expressions for object and logical functions (i.e., the functions themselves), to sign expressions for propositions (i.e., the situations described by propositions), etc. Nominal definitions are used to define the various types of sign expressions themselves either irrespective of their meanings (as for instance in abbreviative definitions) or with reference to their meanings.

In logic, the structure of an explicit (see 1.4) definition (nominal or real) is usually described by one of the following expressions, according to the authors' preferences:

$Dfd = {}_{Df}Dfn$; $Dfd =_{Df} Dfn$;
$Dfd \sim {}_{Df}Dfn$; $Dfd \leftrightarrow Dfn$; $dfdDdfn$.

Each of them reads: 'Dfd equals Dfn by definition.' These general schemata for explicit definitions have been introduced in monographs and manuals on mathematical (symbolic) logic. They are oriented at those types of explicit definition which appear in formalised languages and formal systems.

Definitions in such systems are characterised by the following features:

1. Rigorously constructed systems necessarily include a 'minimal vocabulary', i.e., a set of elementary or basic terms undefined within the framework of the theory and a set of corresponding objects, if the theory has a certain interpretation from the outset. In formal systems these terms are sometimes regarded as unspecified or 'abstract' objects. Therefore, in extending a formal system (e.g., a formalised axiomatic theory), explicit definitions only serve to introduce new terms (and the corresponding objects). Naturally, the new terms and the corresponding objects are introduced into the theory 'by definition'. By means of the definition and on its basis, a convention is established as regards introduction of a new sign expression instead of the one previously used in the system. To be more precise, it is only

[1] Predicators are signs for properties and relations (i. e., for predicates.

through definition that such an object comes under consideration.

2. In such cases, terms or objects are as a rule introduced at one level of discourse: terms are defined by other terms, and objects, by other objects (e.g., functions are defined by other functions). For some analogues of semantic definitions ('A pentagon is a polygon having five sides'), where the Dfd is a sign expression and the Dfn a corresponding object, formal systems use, for convenience, a special sign—the sign of graphic equivalence. Thus, in the theory of algorithms the expression $A \overline{\circ} 1+2$ means that A is merely an abbreviation for a 'word' in a certain alphabet consisting of three letters: 1, +, 2.

It is advisable to retain this sign in *meaningful* semantic definitions as well, bearing in mind that it will not be used here for the sake of temporary convenience, and that ordinary nonformal semantic definitions are an important means of extending the languages of science.

In ordinary scientific and linguistic activity, we usually deal with a broader range of definitions performing new functions. Thus, we often have to (1) specify the meanings of terms that are already current in ordinary language or in scientific usage; (2) record in an explicit definition the intuitively comprehended meaning of a term that has newly been introduced into some language; (3) use semantic definitions to introduce new terms essentially extending a language.

In the first two cases, the term and the corresponding object are not introduced in a meaningful theory or natural language through a definition: both pre-exist definition. Therefore the reading of the sign expression $\text{Dfd} =_{\text{Df}} \text{Dfn}$ as 'Dfd equals Dfn by definition' is at variance with the functions of definitions introduced, if one has in mind that meaning which one reads into this sign expression in the systems and theories of mathematical logic, where new objects or new sign expressions are always introduced within the theory by definition. This descrepancy is manifested most clearly when we deal with a real definition of a material object: it pre-exists any definition, is not introduced by it but merely specified on the basis of the definition in accordance with certain requirements, although for some types of real definition the expression 'by definition' reflects the true essence of the matter (see 4.5 and 7.2). Moreover, the equality of the Dfd and Dfn by definition is not always obvi-

ous. Thus, in semantic meaningful definitions, where the Dfd is a certain term and the Dfn is an object, this equality is not immediately apparent; on the contrary, it is intuitively obvious that the object is never equivalent to the sign expression which names it. In abbreviative definitions in formal uninterpreted systems, objects are identified with each other although they have quite different sign expressions.

Despite all this, however, it is advisable to retain the sign $=_{Df}$ ('equals by definition') to express the general structure of explicit definitions in meaningful theories, in ordinary languages, and in mathematical logic—with certain specifications. We shall therefore read a different meaning into the expression 'equals by definition', namely: 'Any definition contains (sometimes implicitly) a means of identifying the Dfd and the Dfn irrespective of their status of existence.' Let us note that 'equality by definition' is not equality (identity) in the proper sense, but it generates this sort of equality in the sense that in a complete definition it may be replaced (on the basis of additional reasoning and additional procedures) by the ordinary equality relation (Dfd = Dfn) or the equivalence relation (Dfd $\sim$ Dfn). Thus, Dfd and Dfn are different sign expressions in abbreviative definitions in formal systems. However, we may identify them, since they may be reduced to one and the same terminal definiens representing an identical combination of elementary symbols (see 3.5).

For convenience sake, we shall use the sign $\equiv$ instead of the sign $=_{Df}$. As for the relation of equivalence, that is, a relation possessing the properties of reflexivity, symmetry, and transitivity (an equality-type relation), we shall denote it by the sign $\sim$. Hence the sign $\equiv$ will be called the sign of *definitional equality* (identity) [1], as distinct from the ordinary sign of equality (identity).

To distinguish between nominal and real definitions, it is expedient to specify the description of their structure. For this one should distinguish between the use of the Dfd and Dfn terms in the functions of *mentioning* and *using*. We suggest the following forms for recording real and nomi-

[1] H. Curry. *Foundations of Mathematical Logic*. McGraw-Hill. N. Y., 1963. p. 188.

nal definitions:

(1) $\underset{\sim}{Dfd} \equiv \underset{\sim}{Dfn}$

(2) $\text{'}\underset{\sim}{Dfd}\text{'} \equiv \underset{\sim}{Dfn}$

(3) $\text{'}\underset{\sim}{Dfd}\text{'} \equiv \text{'}\underset{\sim}{Dfn}\text{'}$.

Underscoring the expressions Dfd and Dfn means that they are employed in the use function. The expressions Dfd and Dfn in quotes mean that they are used in the function of mentioning. Notation (1) expresses the structure of a real definition as the terms Dfd and Dfn are employed in the use function. Formulas (2) and (3) express the structures of nominal definitions as the term Dfd is here used in the function of mentioning. Notation (2) expresses the structure of semantic nominal definitions; it could be written like this: Dfd $\underset{\sim}{\equiv}$ Dfn. Notation (3) covers two kinds of nominal definition: those in which the semantic equality of the Dfd and Dfn terms is established, and those in which the Dfd term is an abbreviation for the Dfn term (definitions of the latter type are sometimes written as Dfd $\rightleftarrows$ Dfn). In formal systems and in the formalised languages of logic the expression Dfd $\rightleftarrows$ Dfn means that the definition acts as a kind of rule permitting the interchange of the Dfd and the Dfn.

Let us note that when an explicit definition is viewed only as the rule for interchanging the Dfd and the Dfn, it does not matter whether the Dfd and Dfn terms are employed in the functions of using or mentioning, that is, whether they are nominal or real: the interchange rule holds for all correctly constructed definitions, and it is unconditionally applicable when the Dfd and Dfn terms are simultaneously *used* or simultaneously *mentioned*. This rule is not immediately applicable to notations like (2). For the rule to become applicable, the notation has to be transformed into either (1) or (3) (see 1.2).

What is the relation between nominal and real nonformal (meaningful) definitions?

Taking definitions to be sentences occurring in a certain language, each real definition may be transformed into a nominal one (cf. 'Commerce is buying and selling goods and other objects for profit' and 'The term "commerce" is used to denote buying and selling goods and other objects for profit'). Similarly, every nominal definition (apart

from nominal definitions that are mere abbreviations, and on condition that one does not take into account the meanings of the sign expressions Dfd and Dfn) may be transformed into a real one.

The choice between nominal and real definitions is in each particular case determined by the goals and tasks of the study, by didactic reasons, by the principles of simplicity, naturalness and other pragmatic attitudes of the subject introducing the definition.

In defining those Dfd's for which there are sign expressions already existing in the language, we have recourse to real definitions or analytic nominal ones (see 1.3). On the other hand, when a new term is introduced (which results in extending the language), one has to use nominal synthetic definitions (see 1.3). Thus, the definition of the phrase 'to solve a new problem' ('A new problem is considered to have been solved') may be realised in the form of a real definition: 'To solve a new problem means to reduce its solution to those problems which we can solve' (cf. 'A new problem is considered to be solved then and only then when its solution can be reduced to the solution of problems that we are capable of solving already'). The corresponding nominal semantic definition would sound unnatural (cf. 'The expression "to solve a problem" will be taken to denote...').

Very frequently new sign expressions are introduced into formalised theories and formal systems through nominal abbreviative definitions, as abbreviations for already existing expressions, but later, when their interpretations are found, they are reformulated as real ones.

It is also frequent that a definition, constructed as a real one at the pre-theoretic level of knowledge, is later reformulated at the theoretic level as a nominal one, extending the language of the theory, and then translated back into a real one.

Determination of definitions by reality and by past experiences takes diverse forms. Real definitions specifying a certain object for which there already exists a term in a certain language, do not differ greatly from ordinary propositions in that aspect which interests us here: their content is a function of the reality they reflect. Here the information that was obtained in the study of the objects designated by the Dfd term is made more precise, correct and explicit through a deeper study of the objects and through

bringing to the fore that part of the original information which is most important and essential for the solution of the problems under consideration.

The languages of scientific theories are usually extended and enriched through introduction of new terms by means of nominal semantic definition (Dfd ≡ Dfn). For example, a new name is introduced for a newly discovered object or phenomenon described through its objective characteristics. In a case like this, the Dfn is some objective information which is a reflection of reality, while the Dfd term is introduced by convention. The introduction of a new term is sometimes determined rigorously enough by the rules for constructing terminological systems acceptable within a certain scientific discipline (cf. the language of organic chemistry). Definitions introduced in this way may then be translated into real ones, where the Dfd term will not be employed in the function of mentioning but in the use function (*Dfd* ≡ Dfn) (justifiability of such transpositions is discussed in 1.2).

Of greater methodological and epistemological complexity are those cases where there is simply no name in the language for the object to be defined and, moreover, the object itself as the *object of cognition* is introduced by means of a definition.

Let us cite just two examples of this. First we have a situation in which the meaning of the term introduced by a semantic nominal definition is determined by the mode of analysing or constructing the reality studied. The introduction of the definition is first and foremost determined by pragmatic reasons. Such cases are frequent when units of measurement are introduced into a science. The introduction of a certain unit of measurement (e.g., foot or metre) is motivated by pragmatic motives. Definitions of this sort (whether they are formulated as nominal or real) are determined by reality in a different way from the definitions of the Moon, iron, the atom, etc. (see 4.5 and 7.2).

Second, there is the situation of definitions introducing such abstract and idealised objects which have, at best, only extremely remote correlates in material reality. The introduction of such objects is conditioned, first and foremost, by the internal needs of abstract theories (these questions will be further discussed in 6.3).

1.2. SEMANTIC AND SYNTACTICAL DEFINITIONS

Definitions are usually divided into semantic and syntactical.

Semantic definitions are taken to mean definitions of the meanings of sign expressions through explicit indications to objects described by their distinctive features. For instance: 'The word "pentagon" is used to denote a flat polygon having five sides'; 'The expression "decaying oscillations" is used to denote oscillations whose amplitudes (resp. energy) decrease with time'; 'The sign h is used to denote an action quantum'.

If the detailed conception of the structure of nominal and real definitions is to be accepted (see 1.1), the structure of the semantic definition will have this form: 'Dfd' $\overset{\sim}{=}$ *Dfn* or Dfd $\overset{\sim}{\sim}$ Dfn (the Dfd term that is mentioned denotes an object where the corresponding Dfn term is used).

The obligatory requirement applied to all complete explicit definitions, namely, the requirement of interchangeability of the Dfd and the Dfn, is not directly applicable to definitions of this structure. As regards real definitions, the interchangeability of the Dfd and the Dfn results from the equality of their extent, their extensional identity. With regard to other types of nominal definition this interchangeability is ensured by the extensional identity of the Dfd and Dfn terms mentioned. In purely formal definitions, where the Dfd is viewed as an abbreviated notation for the Dfn, this sort of interchangeability is guaranteed by a reduction procedure based on the so-called defining axioms (see 3.5).

In the Dfd $\overset{\sim}{\sim}$ Dfn scheme given above, the sign $\overset{\sim}{\sim}$ is essentially a sign for the designation relation. This relation presents considerable difficulties for logical analysis.

To ensure the applicability of the rules of interchangeability to semantic definitions, they are usually transformed into real definitions or nominal ones, having respectively the *Dfd* $\overset{\sim}{=}$ *Dfn* or 'Dfd' $\overset{\sim}{=}$ 'Dfn' structure.

Thus, the semantic definition cited above may be transformed into definitions of the types: 'A pentagon is a flat polygon having five sides' (a real definition); 'The term "pentagon" has the same meaning as the term "a flat polygon having five sides"' (a nonsemantic nominal definition).

To what extent are such transformations justified?

Semantic definitions always contain a description of a certain object Dfn (the Dfn term is in this case employed in the use function) for which a name is introduced (the term is employed in the function of mentioning). The $\overline{\bigcirc}$ relation is a designation relation.

Two cases may present themselves here: first, a semantic definition introduces a new Dfd term that does not exist in the language, and second, a sign expression already existing in the language is considered with a view to specifying or explicating it. In the latter case it will be interpreted like this: the sign expression defined is used *for the sole purpose* of expressing the information contained in the description on the Dfn properties and for designating objects corresponding to this description.[1] One ignores other information previously associated with the sign expression defined.

The above may serve as the basis for formulating a postulate establishing the condition of equality of certain contexts with regard to their semantic information. *Two contexts that are explicit definitions, each of which is obtained from the other by means of a translation consisting in replacing sign expressions employed in the function of mentioning by appropriate sign expressions employed in the function of using, and vice versa, we shall regard as carrying equal semantic information*, the replacement being effected in a manner to produce definitions of one of the following forms:

$$Dfd = Dfn; \quad \text{`}Dfd\text{'} = \text{`}Dfn\text{'}; \quad \text{`}Dfd\text{'} = Dfn.$$

Thus the following definitions carry the same information: '*h* denotes an action quantum' (a nominal semantic definition); '*h* is an action quantum' (real definition); 'The expression *h* has the same meaning as the expression "an action quantum"' (a nominal nonsemantic definition). In a similar way we shall take it that the following two sentences: 'The first spaceman was Yuri Gagarin', 'The name of the first spaceman is Yuri Gagarin' carry identical semantic information.

The following theorems may be derived from this postulate as direct corollaries:

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[1] In other words, the sign expression defined may be regarded both as a name for the properties indicated in the Dfn and as a name for the corresponding class (see R. Carnap. *Meaning and Necessity*. Univ. of Chicago Press, Chicago, 1956, § 26).

T. 1. A semantic definition of the form Dfd $\overline{\bigcirc}$ Dfn is equivalent, from the point of view of semantic information contained in it, to a real definition *Dfd* = *Dfn*. This may be written as follows:

Dfd $\overline{\bigcirc}$ *Dfn* $\rightleftarrows$ *Dfd* = *Dfn*.

T.2. A semantic definition of the form Dfd $\overline{\bigcirc}$ Dfn is equivalent, from the point of view of semantic information contained in it, to a corresponding nominal nonsemantic definition '*Dfd*' = '*Dfn*':

Dfd $\overline{\bigcirc}$ *Dfn* $\rightleftarrows$ '*Dfd*' = '*Dfn*'.

T.3. A real definition of the form *Dfd* = *Dfn* is equivalent, from the point of view of its semantic information, to a nominal definition of the form

'*Dfd*' = '*Dfn*': *Dfd* = *Dfn* $\rightleftarrows$ '*Dfd*' = '*Dfn*'.

Explicit syntactical definitions in the broad sense are interpreted as definitions which may be directly (i.e., without preliminary translations) regarded as rules for interchangeability of the Dfd and the Dfn in any standard contexts.[1]

The schema for syntactical definitions is frequently written thus: Dfd $\rightleftarrows$ Dfn.

The concept of syntactical definition in the broad sense acquires a precise meaning only with regard to formal systems and formalised languages, where explicit definitions are used to introduce a new symbol (or combination of symbols) with the aid of the sign $\rightleftarrows$ to replace some other symbol or combination of symbols. This new symbol Dfd is regarded as an abbreviation of a certain combination of symbols already present in the system. Such definitions extend the language of the system and are simultaneously regarded as the rules for substituting the Dfd for the Dfn and vice versa. This rule may be substantiated in a general form for this type of systems (see 3.5); at any rate, there is a general algorithm for such a substantiation.

Meaningful (nonformal) definitions of the type *Dfd* = *Dfn* and 'Dfd' = 'Dfn' may also be directly regarded as rules for interchangeability of the Dfd and the Dfn (we may ignore other functions of definitions for the time being).

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[1] The standard context concept will be specified in 2.3.

This substantiation, however, is given on the basis of analysing the concrete content of definitions.

Syntactical definitions in the broad sense are contrasted with semantic definitions ('Dfd' = *Dfn*) which do not admit of direct interchangeability of the Dfd and Dfn. Such interchangeability only becomes possible after they have been translated into corresponding nominal or real definitions.

The syntactical definition is sometimes interpreted in the literature on logic as a kind of real definition where the Dfn is a description of the defined object through those procedures and operations that may be performed upon such an object. These definitions will be called *syntactical in a special sense.*

Syntactical definitions in a special sense are opposed to those definitions in which the Dfd is specified through a description of its properties.

Let us illustrate syntactical definitions in the broad and the special senses.

Supposing we are given a complex description 'The secretion of the Langerghans islands of the pancreas'. This description may be abbreviated by the introduction of the simple term 'insulin'. Then the description having the form 'Insulin = the secretion of the Langerghans islands of the pancreas' will be a syntactical definition in a broad sense, since it is interpreted as a rule for the interchangeability of the Dfd and the Dfn.

In the various games (chess, cards, etc.) the values of the pieces or cards are determined by their roles, by the permissible rules for manipulating them during the game. The names of the pieces (the king, the queen, the castle, etc.) in chess may be viewed as abbreviations for describing their initial positions on the chessboard at the beginning of the game and the possible moves in the game. The pieces cannot be defined through their properties: generally speaking, all of them may be of varying form and made of different material (for instance, having lost the king, we may substitute a button, a lump of sugar, a pebble, etc., for it). In this case we shall be dealing with syntactical definitions in a special sense.

Of a similar kind are the definitions of the natural numbers series in constructivist mathematics and logic through

the description of the procedure generating the number sequence 1, 2, 3, 4, 5... .

If, for instance, the natural numbers series is regarded as previously given or constructed (e.g., generated from the empty set according to Frege and Russell), we may define the various numbers by describing rules for operating upon them. Thus, the description 'The number, greater than zero, which, being added to itself, yields its own square' may be replaced by the sign '2'. We shall then have the definition: '2=the number, greater than zero, which, being added to itself, yields its own square.' Such objects as parentheses and punctuation marks are defined in a similar way, that is, through the rules of using them, of manipulating them in different contexts. Of a similar type are, for instance, the definition of a vessel as something suitable for pouring in a fluid for storing, boiling, etc., the definition of food as something that may be eaten to satisfy hunger and sustain life; the definition of chairman at a meeting as a person performing certain functions as organiser, etc.

The so-called operational definition should be viewed as a kind of syntactical definition in a special sense (we shall take them up in Chapter 4 for special discussion). Analysis of semantic and syntactical definitions will be continued in connection with a discussion of definitions in formalised languages and formal systems (see Chapter 3).

1.3. ANALYTIC AND SYNTHETIC DEFINITIONS

A distinction is made in logic and philosophy between analytic and synthetic definitions.

The problem of analytic and synthetic propositions, analytic and synthetic knowledge in general was, as we know, the subject of detailed discussion in Kant's logic and philosophy (and before that, in Leibnitz). According to Kant, an analytic proposition *a priori* possesses the features of universality and necessity and is therefore true irrespective of the evidence of direct experience. Thus, any attributive proposition in which the predicate P is included in the subject S (this fact being known irrespective of any reference to experience) is analytic and therefore true (e.g., 'any body has extension'). Some synthetic propositions may also possess properties of generality and necessity (e.g., '5+7=

12') and therefore be universal truths independent of any experience, but the basis for this independence is quite different ('synthetic propositions *a priori*'). The same problem is also discussed in the modern literature on philosophy, logic, and methodology of sciences.[1]

The problem of the analytic and the synthetic is also essential for definitions. Richard Robinson divides nominal semantic definitions into analytic and synthetic.[2]

This division may be accepted on condition that all real definitions are translated into semantic ones. Translations of real definitions into semantic nominal ones will belong to the class of analytic definitions. One should also assume that nonsemantic nominal definitions of the form 'Dfd' = 'Dfn' are also transposed to the level of semantic ones: there may be both analytic and synthetic definitions among them. It therefore follows that the division of definitions into nominal and real is not isomorphic with their division into analytic and synthetic, since analytic definitions will include certain nominal ones, as well as translations of real definitions into nominal ones, whereas synthetic definitions will constitute only a part of nominal definitions.

Translation of all definitions into semantic ones with the aim of establishing their analytic or synthetic nature is determined by the goals of research, in particular by the need for elaborating a distinction between the various kinds of analytic and synthetic definitions.

In his work 'On Analytic and Synthetic Definitions' [3] L. Borkowski distinguishes three interpretations of analytic and synthetic definitions.

The first interpretation of analytic (resp. synthetic) definitions is determined by the fact that the definition of a certain word *W* is formulated with regard to a *given* language *S*. This interpretation of analytic and synthetic definitions he calls *absolutely valid*. The second interpretation of analytic (resp. synthetic) definitions is determined by the fact that a certain analytic or synthetic definition of the

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[1] Ye. D. Smirnova. 'On the Problem of the Analytic and the Synthetic'. In: *Filosofskiye voprosy sovremennoi formalnoi logiki* (Philosophical Problems of Modern Formal Logic), Akademiya Nauk SSSR Publishers, Moscow, 1962.

[2] R. Robinson. *Op. cit.*, pp. 96-106.

[3] L. Borkowski. 'Über analytische und synthetische Definitionen'. In: *Studia Logica*, Vol. IV. Poznan, 1956.

word W belonging to a certain language S is nevertheless correlated with another language. This interpretation he calls *relative*.

The third interpretation of analytic and synthetic definitions as formulated by Borkowski has to do with the fact that a definition may be correlated with a certain person. This interpretation he calls *pragmatic*.

Analytic and synthetic definitions in the first interpretation may be defined thus:

D is an *analytic* definition of the word W in the language $S \equiv$ 1) D is a definition of the word W in the language S; 2) the language S contains propositions that have been proved true and that in their turn relevantly [1] contain the word W and are propositions proved true regardless of the definition D.

D is a *synthetic* definition of the word W in the language $S \equiv$ 1) D is a definition of the word W in the language S; 2) the language contains no propositions proved true which would relevantly contain the word W and which would not be considered as such except on the basis of the definition D.

Examples of analytic definitions are furnished by definitions of words included in defining dictionaries, since these words form part of many theses of the natural language, their meanings being different from the ones specified by dictionary entries. Supposing the term W occurs in a certain scientific discipline. We then specify it by means of the definition D. This definition will be analytic.

We deal with absolutely valid synthetic definitions in cases where natural language is extended by adding words whose meanings are introduced by definitions that did not previously exist in the language. This has to do with the appearance of new objects, phenomena, and situations previously unknown in reality. That was the way in which the words *sputnik* or *cosmonaut* appeared in the language.

Supposing that in systematically constructing Euclidean geometry (language S) we use definitions to introduce for the first time the terms 'circle' or 'square'; these defini-

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[1] The word W is relevantly contained in a proposition proved true if and only if its replacement by a constant word of the same semantic category, belonging to the same language and differing from the word W in its extent, results in the proposition becoming neither proved nor true.

tions will be synthetic. These terms may then recur in the theory many times, but their original definition will still be synthetic. New terms denoting newly discovered elementary particles of the microworld are also introduced through synthetic definitions.

Let us illustrate analytic and synthetic definitions in the relative sense. A definition introduced by a teacher when instructing pupils may be synthetic in the pupils' 'language' but analytic in the teachers' 'language'. An explicit definition of disjunction in the axiomatic propositional calculus (language S), where the sign $\rightarrow$ (the implication sign) and the sign $\nearrow$ (the negation sign) appear as elementary terms with regard to the given language S, is a synthetic definition. This definition, however, is analytic with regard to language S_1—a meaningful algorithmic construction of logic where disjunction is defined by means of a table. The definition of the phrase 'a natural number' will be synthetic in set theory and in logic, but analytic in regard to the language of ordinary arithmetic.

The absolute and relative interpretations of analytic and synthetic definitions with respect to a certain person P presuppose that the latter expresses a certain judgement concerning the definition. If the word W to be defined appeared in other contexts in his experience or his language, he will class this definition as analytic, which may be manifested in his evaluation of this definition as a correct one, as one in which the rule for the interchangeability of the Dfd and the Dfn has been observed. If the person P cannot produce such evaluations, that means that the definition is synthetic for him. In taking the pragmatic approach to the analysis of the definition, one realises that the person's evaluation of a definition as analytic or synthetic is really determined by the objective facts of his general training, former experiences, etc. (evaluations of definitions by different persons will vary with these factors). However, his evaluations concerning the truth or correctness of definitions may be of epistemological nature and be independent of psychological factors.

Thus an *analytic definition* in the most general sense is an explicit formulation of the meanings of terms already existing in a certain language, where they may be originally defined irrespective of the definition introduced, e.g., implicitly (contextually).

A *synthetic definition* in the most general sense is the definition of the meaning of a term newly introduced into a certain language or a conscious specification or change in the meaning of a term already existing in some language; in this last case a new meaning is introduced for a term already in existence in some language. Let us note that the interchangeability rule for the Dfd and the Dfn based on the identity of their extensional characteristics has to be checked only in regard to analytic definitions. As regards synthetic definitions, the correctness of this rule is guaranteed by the construction of the definition itself, which may be directly and naturally interpreted as an abbreviative definition (we do not take into account, of course, the historical limitations on our knowledge).

Analytic and synthetic definitions are nominal in as much as the definition of a term and not of the corresponding object is involved (the Dfd term is mentioned in these definitions). Let us note that all real definitions may be viewed as translations (see 1.2) of corresponding analytic definitions. That is why real definitions are here regarded as those in which the object is already represented by some name in some language before definition.

Terms that are introduced into a language through synthetic definition sometimes become analytic in nature at an advanced stage in the analysis, in their explication in a new definition.

1.4. EXPLICIT AND IMPLICIT DEFINITIONS

Explicit definitions are those in which the Dfd and Dfn terms are both present (their generalised scheme: $\mathrm{Dfd} =_{\mathrm{Df}} \mathrm{Dfn}$ or $\mathrm{Dfd} \equiv \mathrm{Dfn}$).

These definitions, if certain requirements are observed, contain both introduction rules (the Dfn term may be replaced by the Dfd term in the most varied standard contexts of the theory) as well as elimination rules (the Dfd term may be replaced by the Dfn term in various standard contexts).

Some types of definition may take the form of explicit definitions—nominal, real, semantic, syntactical, analytic, synthetic, etc.

Explicit definitions are divided into *absolute* and *nonabsolute*. The term *absolute* is applied to definitions in which no

part of the Dfd having independent lexical meaning appears in the Dfn, whether they are employed in the function of mentioning or that of using. In correctly constructed definitions we deal with cases like this whenever the Dfd consists of one word (cf. the definitions of man, function, insulin, etc.). The term *nonabsolute* is applied to definitions in which a certain part of the Dfd having an independent constant lexical meaning is included in the Dfn term. For example, 'Decaying oscillations are those oscillations whose amplitudes (resp. energy) decrease with time'. The word 'oscillations' is here used both in the Dfd and the Dfn. Such definitions have the form: Dfd(A, B)$\equiv$Dfn(...B...) (the Dfd and the Dfn terms both include B, but the complex term Dfd is constructed of A and B).

The definitions of prime number, even number, virtuous person, recursive function, etc., are of a similar type. Such repetitions of parts of the Dfd in the Dfn do not imply a vicious circle. We usually deal with nonabsolute definitions in cases where certain classes of objects (e.g., numbers or functions) and the corresponding general names already have definitions at the scientific level or that of ordinary language. Here the problem arises of defining certain well-formed subsets of this class (the elements of these subsets may prove to have specific properties significant for cognition or practice). If these subsets do not have specific absolute names in the language, we may use a definition to introduce a new complex (nonabsolute) name in such a manner that it will indicate the part of the already familiar class to which it refers (it should be clear from the structure of the name that even numbers are a subset of the natural numbers, and recursive functions, a kind of functions). The vicious circle does not arise in nonabsolute definitions as they define a complex name that has a new meaning as compared to the meanings of the terms A and B, and not the meaning of the term B recurring in the Dfd and Dfn terms (as well as the meaning of the term A). This complex name may, generally speaking, be replaced by a new absolute name, so that nonabsolute definitions will take the form of absolute ones.

Definitions of predicates presented as propositional functions, such as: the prime number $(x) \equiv x>1 \wedge \forall y\ (y\neq 1 \wedge y/\!/x \rightarrow \equiv x)$, are not regarded as nonabsolute despite x being part of both the Dfd and the Dfn. The fact is that the free variables are not terms having independent lexical meanings.

In definitions of this kind the meanings of terms are defined, namely, those of predicates, but this is not done through reducing the latter to the domain of the values of x which they satisfy (i.e., to the extensional of the function 'x is a prime number'), but through other predicates.

Implicit definitions will be interpreted as definitions through axioms in axiomatic theories (let us call them implicit definitions in the proper sense). Other kinds of definition characterised as implicit but capable of being transformed into explicit ones according to certain algorithms will be treated specially. For implicit axiomatic definitions there are no rules for realising this transformation within the theory: the elementary terms of the theory are mutually defined here in contexts that cannot be eliminated within the framework of the theory.

As has already been pointed out, implicit definitions (in the proper sense) are axiomatic definitions, i.e., definitions of elementary objects through elementary propositions (axioms) in axiomatic theories.

The specific features of implicit definitions as compared to explicit ones are most clearly manifested in regard to the following points.

As distinct from explicit definitions of the Dfd $\equiv$ Dfn structure, implicit definitions do not include the Dfd and Dfn concepts; as a result, they are not identified in any way. Implicit definitions are used to single out a certain system of sets having a certain structure in such a way that application of rules of logic to an axiomatic description may yield new information about these sets and enable one to prove theorems about them. The elementary objects of the system (and the corresponding terms) are mutually defined: implicit definitions are circular definitions, but their circularity is not vicious. As these definitions are circular, each elementary object (resp. term) cannot be defined through others: an implicit definition of a certain object (or term) cannot be replaced by a corresponding explicit definition. That means that there are no *elimination rules* for implicit axiomatic definitions within closed theories. As for explicit Dfd $\equiv$ Dfn definitions, there are not only *rules for the introduction* of objects (and corresponding terms) but also *rules for their elimination* within the theory. Thus we may substitute the Dfd term for a certain description corresponding to the Dfn (an introduction rule). Then, as the need

arises, we may eliminate a certain Dfd term in a certain context replacing it by a Dfn term (an elimination rule; see Chapter 8).

Algebra furnishes examples of the simplest axiomatic definitions. Let us cite the algebraic definition of group as an illustration.

A nonempty set G is called a group if an algebraic operation is defined for it (conditionally called multiplication) which establishes a correspondence between each pair of elements a and b from G and another element ab also from G conditionally called their product and having the following properties:

(1) $a(bc)=(ab)c$ (the law of associativity).

(2) For each a and b from G the equations $ax=b$ and $ya=b$ are decidable in G, that is, there are elements c and d in G such that $ac=b$ and $da=b$ (the law of reversibility). [1]

This definition is valid for certain sets with algebraic operations defined on them, which are for this reason, by definition, groups. Of this nature are all integers, all rational, real, and all complex numbers with regard to the operation of addition of numbers, having the same function as the group operation of multiplication. None of these sets are groups with regard to the operation of multiplication of numbers, since equations of the form $0 \cdot X=1$ have no solution. But all rational, all real and all complex numbers excepting 0 are groups with respect to the operation of multiplication. This definition may thus be used to single out assemblages of concrete sets with corresponding operations defined on them, which may be described as groups.

This axiomatic description introduces a certain system of interconnected objects beyond the limits of the axiomatic theory, namely, sets with operations defined on them, which satisfy the axiomatic description. The properties of these operations are described by the axioms (1) and (2).

Let us consider the second example.[2]

Let a system of undefined objects consist of the set (or domain) D, zero—an element of this set, and a unary ope-

[1] See *Entsiklopediya elementarnoi matematiki* (Encyclopedia of Elementary Mathematics). Book 1, *Arithmetic*, Gostekhizdat Publishers, Moscow-Leningrad, 1951, p. 102.

[2] S. C. Kleene. *Introduction to Metamathematics*. D. Van Nostrand Company, New York, 1952, pp. 25-27.

ration on the elements of the set D, i.e., the operation ($'$). This system of objects will be denoted by the expression $\langle D, 0, ' \rangle$.

This system of objects, just as in the previous example, is an abstract one, as we know nothing about the nature of the objects apart from correlations between them described by the relevant axioms. An axiomatic system describing the relations between them is a meaningful one, as the description is in ordinary (nonformalised) language. Such axiomatic systems are frequently called abstract meaningful ones. On the other hand, when the same axiomatic systems are opposed to concrete meaningful systems (cf. the axiomatic presentation of geometry by Euclid), where the objects described by the axioms are believed to be familiar before the axioms are introduced, they are often called formal.

If the expression 'is a natural number' (i.e., a certain fixed predicate) in Peano's axiomatic system is to be replaced by the expression 'an element of a certain set D', we shall obtain the following system of axioms:

A.1. $0 \in D$.

A.2. If $n \in D$, then $n' \in D$.

A.3. If $m \in D$ and $n \in D$, then $m' = n'$ only if $m = n$.

A.4. If $n \in D$, then $n' \neq 0$.

A.5. Let $P \subseteq D$, P having the following properties: (1) $0 \in P$; (2) if $n \in P$, then $n' \in P$; then $P = D$.

A.5 is nothing else but the principle of complete mathematical induction. It is easy to see that the natural number series satisfies the requirements of this axiomatic system: if D is interpreted as the set of natural numbers, 0 is interpreted as the zero number, and the '' operation as 'following x', all axioms will be true propositions.

There may be other models or representations of this theory that will satisfy it (not only mathematical ones). All these representations, however, will be isomorphic with the set of natural numbers. Two representations (sets) are called mutually isomorphic if a 1:1-correspondence may be established between the elements of the first set and the elements of the second set and the predicates defined on them have the following property: if they are valid (or not valid) for certain fixed elements of the first set, they are correspondingly valid (or not valid) for the elements of the second set that have been paired (in establishing a 1:1-

correspondence) with the elements of the first set, and vice versa.

If systems of objects are mutually isomorphic and one of them satisfies the requirements of an axiomatic system, the others also satisfy them.

If two systems of objects are not mutually isomorphic and one of them satisfies some axiomatic system, two alternatives are possible: (1) the other system does not satisfy it; (2) the other system does satisfy it.

Supposing we are given a system of module 2 subtractions: 0, 1, 0, 1, 0, 1, ...

The natural number sequence will be transformed into this system if each number beginning with zero is replaced by the remainder of dividing it by two.

This system of objects is not isomorphic with the natural number sequence, since the 1:1-correspondence cannot be established between these systems of objects. We deal with case (1) here: it does not satisfy our axiomatic theory (in particular, axiom A.4 does not hold). However, if we take the system of axioms A.1-A.3, A.5, both the natural number sequence and the module 2 subtraction system will satisfy it. We deal here with case (2). Finally, we may also come across axiomatic theories that have no representations whatever: those are contradictory axiomatic theories.

We shall be dealing with a theory of this kind if we, for instance, add to the axiomatic system A.1-A.5 the following axiom A.6: if $n \in D$, then $n' \neq n$. but $n'' = n$ (this axiom, by the way, is valid for the module 2 system of subtractions). It might be demonstrated that A.6 contradicts A.4, which makes the whole system of axioms A.1-A.6 contradictory.

Thus, there are three types of axiomatic theories.

1. Those in which all representations are mutually isomorphic. Such systems of axioms are called *categorical*.

2. Those having representations that are not mutually isomorphic. These systems of axioms are called *incomplete*.

3. Those for which there are no representations. These systems of axioms are called *unrealisable*. As distinct from unrealisable systems, categorical and incomplete systems are called *realisable*.

We shall now discuss this question: in what sense can abstract meaningful axiomatic systems be regarded as definitions? This seems to be important as not all logicians are inclined to regard axiomatic systems as a special kind of

definitions, that is, to view them as so-called implicit definitions; they are sometimes considered to be a specific mode of introducing the primary objects of a theory, which is done simply by means of axioms, not definitions.

When concrete meaningful axiomatic systems were constructed (cf. the construction of geometry by Euclid), where the objects under consideration were regarded as given and to a certain degree analysed before the construction of the theory, and where explicit definitions of objects under study were introduced before formulating the axioms, the latter being interpreted as true propositions describing correlations between them, the question did not arise in the methodology of science whether axioms were definitions or not. Axioms were unconditionally excluded from definitions.

In abstract meaningful axiomatic systems, as we have seen, concrete objects under consideration and their correlations appear in the form of corresponding variables; within the context of the theory, they are only 'defined' at the grammatical level. For instance, with regard to the system of axioms of group theory, the only thing that we knew of the sign G was that it was a variable for sets, just as we knew of the signs a, b, c that these were variables for the elements of the set G; we knew, further, that the expression 'an operation conditionally called multiplication' was a variable for mathematical operations. In a similar way, as regards the Peano axiomatic system (suitably transformed), the only thing that we knew of the elementary objects (and corresponding terms) was that D was a certain set, 0 was an object belonging to that set, while $'$ was a certain unary operation on the elements of the set D. Since meaningful scientific theories that are not formally constructed presuppose introduction of explicit definitions of objects under study, the question arose whether theories might at all be constructed without definitions or whether the axioms themselves play the role of definitions here.

Most logicians are at present inclined to regard abstract meaningful axiomatic system as *implicit* definitions of a particular type, although it is generally clear that the axiomatic mode of introducing the objects for consideration is essentially different from the explicit definitions mode. We believe that the interpretation of abstract axiomatic systems as definitions is to a certain extent substantiated (although, generally speaking, one might think of a new name

for this mode of introducing objects into science). S. C. Kleene writes in this connection: 'The axioms of an axiomatic theory are sometimes said to constitute an implicit definition of the system of the objects of the theory; but this can only mean that the axioms determine to which systems, defined from outside the theory, the theory applies.'[1]

Quite obviously, it is only possible to use axioms to single out systems of objects that are outside the theory and satisfy this theory, when the context consisting of a set (or conjunction) of axioms possesses a certain semantic meaning independent of the concrete domains of their application.

The addition of each subsequent axiom (if it is independent from others) narrows down the range of its application to objects. Thus, in the Peano axiomatic system A.1 reads that $0 \in D$. That means that 'a certain object 0 is an element of the set D'. Any nonempty sets will satisfy this axiom considered in isolation. The axioms that follow, however, state that the elements of the set D are ordered within a certain sequence, that 0 is the first element of this sequence, etc. That means that *the context of the axioms as a whole determines the structure of the sets which satisfy the axioms.*

Generally speaking, elementary propositional functions which include only variables, e.g., $P(x)$, $R(x, y)$, $R(x, y, z)$, etc., also determine sets for which they are valid, namely, they enable one to distinguish between sets corresponding to one-place predicates, sets corresponding to two-place predicates, etc. However, they are not regarded as definitions in general, since they do not function as definitions in the corresponding systems: they do not contain requirements or rules nor are they viewed as theorems of a system (see 3.1). Each axiom of the theory may be regarded as a theorem—as an explicit definition introduced into the system. Axioms are used in developing theories, in proving theorems, just as explicit definitions extending the range of our deductive possibilities.

It is common that any theory, rigorous or not, includes within its basis the definitions of those primary objects (resp. terms) which it studies, although the modes of defining them may differ essentially. This seems to have been sufficient reason for including among definitions the procedures involved in the axiomatic mode of introducing the

[1] S. C. Kleene. *Op. cit.*, p. 27.

primary objects of a theory, the definitions in question being given no new specific names. The specificity of these definitions is emphasised by the use of the phrase 'implicit axiomatic' in the phrase 'implicit axiomatic definitions' which is the name for the mode of introducing the objects of the theory considered here.

In a certain sense, implicit axiomatic definitions may also be interpreted, in our view, as explicit ones, too. Many implicit axiomatic definitions are frequently formulated as corresponding explicit ones (in a special sense).

If we introduce a special name for objects satisfying axiomatic descriptions, we shall obtain an explicit definition of these objects in the sense that this name may be eliminated within the framework of the theory in the same way as in explicit semantic definitions (the Dfd and the Dfn become interchangeable in different contexts).

Thus, in the case of the Peano axiomatic system (see the second example above), we might introduce the name 'natural number' for the isomorphic systems of objects satisfying the Peano system, that is, a name for the 'elements of the set D' (cf. the definition 'The elements of the set D are called natural numbers if and only if they satisfy the axioms A.1-A.5'). The set of sets of systems of isomorphic objects will be defined here, each of which is a representation of the system of natural numbers.

For systems of objects described by a set of axioms (the axiomatic systems being incomplete, i.e., noncategorical), it is sometimes difficult to think of a general name; compare, for example, the axioms A.1-A.3, A.5 (see p. 38) which are valid for the system of module 2 subtractions and the natural numbers system.

The system of objects of the abstract axioms of Euclidean geometry might be given the name 'the objects of Euclidean geometry'.

Axiomatic definitions are often formulated as explicit definitions in the above sense. The definition of the group considered above ('A nonempty set is called a group if...', see p. 37) was of this kind.

The above explicit definitions emerging from the corresponding systems of axioms are explicit only in the sense of interchangeability of the Dfd and the Dfn within the theory. They differ essentially, however, from other explicit definitions in the Dfn structure, and in the mode of determining

those systems of objects beyond the theory which satisfy them. We therefore suggest that such definitions should be called quasi-explicit axiomatic definitions.

Modern axiomatic theories should be distinguished from intuitive axiomatic theories illustrated by the Euclidean axioms. In Euclid's *Elements* the separate 'primitive' terms of the system are given explicit definitions. These definitions lie beyond the system itself and form a premise, as it were, for its construction. The terms in which the definitions of a number of primitive concepts are described do not appear in the formulations of the relevant primitive propositions (postulates and axioms) or in the proofs of theorems, therefore such terms in Euclidean geometry as 'point', 'straight line', 'surface', 'plane' should be regarded as primitive terms introduced without definitions. In proving the relevant theorems only those properties of the terms are taken into account which are described by the corresponding primitive propositions (axioms and postulates).[1]

The definitions of various primitive terms in Euclid's *Elements* are formulated as real ones and are assumed to be intuitively clear, while the primitive propositions describing the properties and correlations of these objects are often regarded as self-obvious truths in the history of philosophy. The properties and correlations of more complex objects introduced by means of ordinary explicit definitions are described in the theorems, for which proofs are given.

Euclidean geometry was constructed and for a long time interpreted as a theory directly describing the properties of real physical space.

The development of the axiomatic theory enters a new stage in the work of Moritz Pasch, Felix Klein and others, reaching its peak in the brilliant *Foundations of Geometry* by David Hilbert. The formation of new views on axiomatic constructions and the nature of axioms appearing in them was considerably influenced by the discovery of non-Euclidean geometry by János Bolyai and Nikolai Lobachevsky.

The first step in constructing such theories is to choose systems of objects, their properties and correlations, the choice being frequently based on the analysis of relevant meaningful constructions, including axiomatic ones.

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[1] On the difference between the postulates and the axioms of Euclidean geometry see S. A. Yanovskaya, *Op. cit.*, pp. 63—96.

It is precisely because the primitive terms at this stage in the construction of the theory are not bound to any fixed interpretation that the axioms cease to play the role of truths (still less that of self-obvious truths). In Stoll's view, they function as theorems that are viewed within the system as proven propositions concerning certain unspecified objects.[1] In this case the truth of axioms is replaced by their demonstrability at the formal syntactical level, since they may be regarded as theorems of the system.

Axioms cannot be directly considered to be *true* propositions; they are implicit definitions of certain unspecified objects. Axiomatic systems, however, are constructed for the solution of definite scientific problems. That is why scholars always look for a definite interpretation of the system, trying to construct a model that would satisfy an abstract axiomatic system. When a model has been constructed, the axioms become true propositions.[2]

In axiomatising one and the same intuitive meaningful theory, different sets of primitive terms may be selected. Thus, in Hilbert's axiomatic system for Euclidean geometry there are six primitive terms: 'point', 'straight line', 'plane', 'incident', 'between', and 'congruent'. In Pieri's axiomatic system for Euclid's geometry there are two primitive terms, 'point' and 'movement'. It is said that we deal here with different formulations of one and the same theory.

For abstract axiomatic systems as well as for most explicit meaningful definitions there is no algorithm that would enable one to find systems of objects satisfying corresponding descriptions, that is, to list all systems of objects satisfying the given description. Such a description is given on the basis of intuitive meaningful reasoning.

At present an axiom is usually defined as a proposition accepted within the framework of a certain theory without proof. This definition of an axiom is entirely acceptable if the methodology of science is restricted to internal problems of science. However, if the methodology of science takes into account both the internal and external problems of science,

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[1] See R. Stoll. *Sets, Logic and Axiomatic Theories*. Freeman, San Francisco-London, 1961, p. 132.

[2] See Ye. K. Voishvillo. 'The Concept as a Form of Thinking'. *Voprosy filosofii*, No. 8, 1969.

the definition of an axiom given above seems to be too narrow.

Many axiomatic abstract meaningful theories or formalised systems, particularly those of the greatest scientific interest, emerge through formalising the corresponding concrete meaningful theories. S. C. Kleene writes: 'The interpretation motivates the metamathematician in his choice of the particular formal system which he introduces by his definitions. It guides him in choosing the problem relating to the system which he investigates. It may even provide him with essential clues toward achieving the solution of those problems. Only in the final statement and proof of his results is he prohibited (as a metamathematician) from using the interpretation.'[1]

When this approach is taken, abstract meaningful or formalised theories are no longer regarded as propositions taken without proof: they are merely left unsubstantiated within the limits of a closed theory. As for the methodological level, where both 'internal' and 'external' problems of scientific theories are taken into account, they *are* substantiated, although not in a strictly formal manner. Such criteria as 'intuitive clarity', 'obviousness', etc., formulated by the great mathematicians and philosophers of the past, seem to play a certain role in this sort of substantiation.

It would therefore be quite expedient to split the concept of axiom by introducing, apart from the definition of axiom in the narrow logical sense, one that is used in the analysis of internal theoretical problems, also the concept of axiom in the specifically methodological sense—one that is involved in the discussion of external theoretical problems of science.

1.5. DESCRIPTIONS AND DESCRIPTIVE DEFINITIONS

There are different ways of singling out individual objects in logic. The most widely used mode of singling out individuals is the use of proper names (cf. 'the Volga', 'Leonardo da Vinci', 'Yuri Gagarin'). However, the number of individuals is much greater than that of proper names.

[1] S. C. Kleene. *Op. cit.*, p. 64.

There is another way of specifying individual objects (individuals) in the natural language, namely, through fixing the predicate or logical combination of predicates, for which one and only one object exists. Predicates and their combinations fixed in the language and playing the role of specifiers of individual objects are called descriptions, or definite descriptions, to be more precise. We have recourse to descriptions as means of specification in cases where individual objects have constant proper names (cf. 'the Volga' and 'the longest river in Europe') as well as in those cases where the object has no proper name or that proper name is irretrievably forgotten (cf. 'the tallest tree in such and such a garden' or 'the inventor of the wheel'). Descriptions sometimes include many-place predicates and a proper name (cf. 'Pushkin's wife'). Objects and events may also be specified through descriptions of their temporal and spatial coordinates. This mode of specification is applied in exact sciences and has the advantage of eliminating all sorts of ambiguities from description and ensuring the application of mathematical methods to it.

When definite descriptions are used, it is assumed that *there is* an object satisfying them (the descriptum) and that there is *only one* such object.

The following notation is usually used for descriptions: $(\iota x)f(x)$, which should be read: 'That x, which has the property f' (f in this case denotes the predicate or the combination of predicates). In this way the ι-operator (iota-operator) transforms a certain function $f(x)$ into a certain name (term): $(\iota x)f(x)$.

Both proper names and descriptions form part of natural language. Proper names and descriptions function as arguments in propositions that are logical functions. The use of descriptions as arguments of functions has that advantage over the descripti of proper names that they reveal the structure of the descriptum and provide additional information about it.

Supposing we are given the sentence 'Ivan the Terrible reigned before Peter the Great'. Let us write it like this: $R(x_1, y_1)$ (1), where R is the relation 'reigned before', x_1 is Ivan the Terrible, y_1 is Peter the Great. Let us substitute 'the son of Grand Prince Vassily III' for the proper name 'Ivan the Terrible': $x_1=(\iota x)S(x, z_1)$, i.e., x_1 is object x that is the son (S) of Grand Prince Vassily (z_1). By substituting

$(\iota x)S(x, z_1)$ for x_1 in (1), we shall obtain a proposition that has a more complicated internal structure:

$R((\iota x)S(x, z_1), y_1)$.

The ι-operator is most frequently used in mathematics to introduce individual abstract objects. In these cases descriptions must unambiguously describe a certain individual object and nothing but that object. The existence of the descriptum signifies consistency of the description, singularity of the descriptum signifies its completeness.

Individual objects are introduced descriptively through definitions taking the form of descriptions: the description is transformed into the Dfn of the definition, while the role of the Dfd is played by the proper name introduced for the descriptum. Thus the description 'That number which cannot be a divisor in arithmetic' is replaced by the definition 'Zero is that number which cannot be a divisor in arithmetic'. Similarly, the description 'That number which, being added to an arbitrary number n, yields n' will be replaced by the definition 'Zero is that number which, being added to an arbitrary number n, will yield n'.

Supposing we are given the description 'That natural number x which has such a property that $x+x=x^2$ and $x>0$'. Only one number, designated by the figure '2', satisfies this description. The corresponding definition will have the form '2 is that natural number x which has such a property that $x+x=x^2$ and $x>0$'. This definition can also be written like this:

$2 = (\iota x)(x+x=x^2 \wedge x>0)$.

The problem of the formal definition of the ι-operator, of the syntactical difficulties involved in such a definition as well as the problem of defining the operator of indefinite descriptions will be discussed later in connection with contextual definitions (see 1.6).

Descriptive definitions have been given different interpretations in logic and mathematics by different authors. Frege, for instance, held the view that descriptive definitions cannot be used to introduce or construct the object which they describe: if an object does not exist, it cannot be introduced by a definition (Frege pointed out, for instance, that one cannot make a diligent pupil out of a lazy one by definition). One may, of course, create a description like 'that number

which is smaller than two and greater than three', but this description is contradictory since such a number does not exist anyway. According to Frege, one cannot introduce object 0 either, unless such an object exists in reality. Rudolf Carnap in his interpretation of descriptive definitions proceeds from Frege's viewpoint. According to Carnap, before we construct and use descriptive definitions, we must prove that there is an object (and only one such object) which satisfies this definition.

The view expressed by Carnap was called in question by many logicians. It was correctly pointed out that, if this view is accepted, descriptive definitions as a particular kind of definition become meaningless, as the proof (in the constructivist sense) that the object satisfying the description exists, means that it may be given beforehand by name and defined in the usual way, without the ι-operator. To defend his view of descriptions Carnap suggested this way out: it is not necessary to have a constructivist proof (e.g., a direct demonstration) for the existence of an object. One may limit oneself to a weaker proof demonstrating that the assumption of the object's existence does not result in a contradiction. Descriptive definitions then become meaningful, serving as the means for introducing new abstract objects (such as imaginary numbers) into science.

The most interesting and mathematically useful view of descriptive definitions was developed by Hilbert in his work on the ε-operator calculus. According to him, such definitions may be introduced irrespective of whether objects corresponding to the definition exist or not, whether there is one object or more: descriptions may be used to define both individual objects and objects that are elements of a certain class ('those objects that possess a certain property P'). In this case the objects become indistinguishable from the point of view of the property P, they are 'glued together', forming a certain 'abstract object'.

If an object does not exist, it may be introduced by description. Logical analysis of such propositions makes it possible to prove the nonexistence of objects introduced by description.

On the whole, a theory operating with this kind of description becomes hypothetical, since existence or nonexistence of objects introduced by descriptions may remain unproved. The presence of such descriptions in the theory is

not, however, fraught with any sort of danger as their introduction is always subject to the reservation 'if such an object exists'. Hilbert calls his operator the ε-operator (the epsilon-operator) as distinct from the ι-operator, which he also analyses.

To justify the introduction through description of new abstract objects not given in sensual experience, Hilbert formulates the following ε-theorem: if in proving a certain theorem we use an object introduced by a description, by an ε-symbol (the formulation of the theorem itself containing no such ε-symbol), and if this proof may be juxtaposed with another proof in which the ε-symbol is absent, that means that the object introduced by the description (by the ε-symbol) is justifiable and may be used. We may use an unproved premise in proving a theorem (treating that premise as if it were proved) precisely because it may later be eliminated by applying the deduction theorem.

Hilbert held the view that the apparatus of logic and mathematics should only be extended in a way that would permit, *under definite conditions*, its reduction to the old apparatus, the latter being capable of yielding new results.

1.6. CONTEXTUAL DEFINITIONS

Contextual definitions may be of two sorts:

1) definitions in which the meaning of the term is given by a certain context or assemblage of contexts, analysis of such contexts serving as the basis on which the definition may be formulated explicitly;

2) definitions in which establishing the meaning of the term is limited to defining the contexts in which it is used

Definitions of the first type we shall call *contextual definitions in the proper sense*, and definitions of the second type, *contextual definitions in use* or, following Hans Reichenbach, simply definitions in use.[1]

Let us consider the first type of definition. In natural language, one frequently comes across unfamiliar expressions in the context of other sign expressions whose meanings are already known, and we do not have any precise algo-

[1] H. Reichenbach. *Elements of Symbolic Logic*. The MacMillan Company, N. Y., 1948, p. 20.

rithms for establishing the meanings of unfamiliar sign expressions on the basis of analysing the relevant contexts. However, the result of this analysis is amenable to formulation in the shape of explicit definitions of the meanings of unfamiliar sign expressions: they prove to be determined by a definite (frequently unambiguously definite) set of contexts. The meanings of these sign expressions will be said to be defined by the context in the proper sense. Specialists on lexicology frequently analyse this sort of contexts, establishing the meanings of certain sign expressions and entering the corresponding explicit definitions of these sign expressions in phraseological and other dictionaries.

Setting aside the possible homonymy of the sign expressions under analysis, all descriptive contexts (properly concretised) in which the definiendum occurs, frequently become not merely semantically meaningful but also admitting of truth evaluations—when the meaning is defined correctly. The work of the lexicologist, however, is never purely analytical, as he may never be sure that the term under consideration is used in the analysed contexts correctly, that is, in a way in which it is used in certain model contexts. Explicit formulation of the meanings of terms is usually linked with their idealisation and reconstruction. As distinct from contextual definitions in the proper sense, implicit axiomatic definitions (which may also be regarded as a kind of contextual definition if the concept of contextuality is properly defined), as we have seen, cannot be transformed into explicit ones within the language of the given system.

Contextual definitions sometimes define a certain type of semantic meaning, not meaning as such. Let us consider the following example.[1]

Supposing that in reading a certain text we come across the following three sentences:

(1) Tar is a biped animal; (2) It so happens that Tar speaks Russian; (3) Tar smokes a pipe. We have to find out the meaning of the word 'Tar'. We conclude that this is a proper name from the fact that the word is written with a capital letter in sentences (1), (2) and (3). We cannot conclude from

[1] This example is given by Bochenski to illustrate axiomatic definitions. From our viewpoint, this definition is not axiomatic, as it can be transformed into an explicit one in the ordinary sense (J.M. Bochenski. *Die zeitgenössischen Denkmethoden*. Francke Verlag, Bern 1954, S. 92-93).

sentences (1) and (2) that the word 'Tar' is a man's proper name since birds are also bipeds, and birds like parrots may be taught to speak (the sentences given above tell us nothing about the kind of command that Tar has of the Russian language). Sentence (3), however, permits us to conclude that Tar is a man's proper name and not, e.g., a parrot's, since birds do not smoke pipes while Tar does.

We have thus merely established the domain of application of the word Tar, but we have not found the concrete person bearing that name.

Let us note that in research we sometimes do not set ourselves the task of finding the concrete meaning of a certain name but merely attempt to establish the type of its semantic meaning, i.e., the domain of application of the name. In the latter case one may hardly speak of ambiguity of contextual definitions, since our task is to establish the domain of application of an unfamiliar name.

If our analysis of a suitable context yielded an unambiguous definition of the individual bearing the name of Tar, our contextual definition might be formulated as a descriptive one containing the ι-operator (see 1.5): $\mathrm{Tar} \rightleftharpoons \iota x P(x)$. What is said here is that Tar is precisely that x (i.e., that person) that has the property P. To stress that this definite description does not introduce a new object but merely singles it out as the result of analysing context C and with regard to it, this definition may be written like this:

$$Tar \rightleftharpoons C(\iota x P(x)),$$

where C is a variable for contexts in which the word 'Tar' is used.

Contextual definitions of this kind also include implicit definitions of unknown quantities in certain equations. There are cases in algebra, the theory of differential equations, mathematical logic, mathematical linguistics, etc., where there is an algorithm for the solution of equations (and their systems), enabling one to find out whether the system has a solution and to find this solution if it does exist. 'Solution' is here understood as a system of meanings which transforms equations into identities. An equation or a system of equations having solutions determine the meaning of an unknown quantity (quantities). Although the meanings of the unknown quantities are not given in an explicit form, we may nevertheless apply algorithms to

equations of certain types which eliminate the context where variables occur.

In solving an equation, we transform an implicit or contextual definition into an explicit definition of the unknown quantities. Thus, in solving the equation $x+4=6$, where the meaning of the unknown quantity is already defined implicitly or contextually, we define it explicitly (the context being eliminated), namely we find that $x=2$ (the meaning of the expression 'x' is the number 2).

Let us now consider *contextual definitions in use.* Instead of defining a certain term regardless of the context, we frequently define it in the context of other terms, those with which it most frequently co-occurs in scientific or linguistic usage. We thus define the context into which we introduce the term, rather than the term in isolation. These contextual definitions are called definitions in use. They are explicit definitions.

Thus, instead of defining the meaning of the term 'the science of logic', we may, for certain purposes, define the meaning of the expression 'to master the science of logic' in this way, for example: 'To master the science of logic means to know and be able to use the rules of reasoning that are described in the relevant textbooks.' Instead of defining the meaning of the word 'wife', we may define the context 'to become a wife'.

Reichenbach gives the following example. To define the term 'metabolism' is difficult enough, but it is even more difficult to use this definition in many practical situations, for instance, when one desires to find out the conditions for normal metabolism and for deviations from the norm. In this case one may restrict oneself to definitions of the contexts 'the metabolism is normal', 'the metabolism is subnormal'.[1]

The value of definitions in use as distinct from definitions *per genus proximum et differentiam specificam* consists in the following. In definitions of the latter type, the less general and the more concrete is always defined through the more general and the less concrete, i.e., through the more abstract (cf. 'man is an animal...', '0 is a number...'). When the results of cognition are being systematised, such an approach is quite justified. However, in the process of cognition

[1] H. Reichenbach. *Op. cit.*, p. 22.

itself or in the more complicated cases of applying the results of cognition to concrete facts, it is expedient to resort to definitions in use. The reason for that is that in describing the process of cognition, the more abstract and more general entities are verified through the less abstract and the less general. These definitions, in our view, prove to be extremely helpful when we attempt to explicate the most general concepts used in most diverse systems and languages. Of this nature, for instance, are categories. The natural mode of defining them is through axiomatic definitions, that is, through describing their correlations (cf. the definitions of essence and phenomenon, form and content, etc.). As for the unpaired categories, it is more natural to resort to contextual definitions in use.

Definitions in use may be interpreted both as nominal and as real ones. They are widely applied in logic and mathematics. Thus, to define classes in terms of functions in logic and set theory, we resort to contextual definitions in use. Instead of defining the class F (or $\hat{x}F(x)$) as a certain set of objects for which the function $F(x)$ has a truth value (where F is some property of objects), we may replace this definition by a more operational definition of the context 'x belongs to the class $\hat{x}F(x)$'. The definition then takes this form:

$$x \in \hat{y}F(y) = F(x). \qquad (1)$$

This proposition defines simultaneously a class and the relation of membership in the class. As distinct from the definition of a class as a set, this proposition demonstrates clearly in what way classes may be eliminated by using functions.

In the above, we have used the ι-operator (the definite description operator 'that ... which'). It may be defined through contextual definition in use.

Let us consider an example. Supposing the sentence (or context) is given: 'M. I. Kutuzov was born in 1745.' Let us write this sentence in symbolic form: $P(x_1, y_1)$. (1), where P is the relation 'to be born in', x_1 is M. I. Kutuzov, y_1 is 1745.

In this sentence we may replace the proper name 'M. I. Kutuzov' by a definite description 'the Supreme Commander

of the Russian Army in the battle of Borodino'. Then

$$x_1 = (\iota x)\, S(x,\ z_1), \tag{2}$$

where S is the predicate 'to be Supreme Commander of the Russian Army in', and z_1 is 'the battle of Borodino'. Let us now substitute expression (2) for x_1 in expression (1). We shall obtain the expression

$$P\,[(\iota x)\, S(x,\ z_1),\ y_1]. \tag{3}$$

This expression may be read as 'That x (man), who was the Supreme Commander of the Russian Army in the battle of Borodino, was born in 1745'.

Let us now define expression (3) through contextual definition in use. We shall write only the Dfn of this definition:

$$\exists x\,\{S(x,\ z_1) \wedge P(x,\ y_1) \wedge \forall u\,[S(u,\ z_1) \rightarrow (u = x)]\}. \tag{4}$$

Expression (4) may be read like this: 'There exists such an x (man) who was the Supreme Commander of the Russian Army in the battle of Borodino and was born in 1745, and there is only one such x.' The singularity of the existence of such a person is established in the expression

$$\forall u\,[S(u,\ z_1) \rightarrow (u = x)]. \tag{5}$$

It is stated here that any u satisfying the function $S(u, z_1)$ will coincide with x.

Thus expression (3) containing a definite description operator may, on the basis of this definition, be replaced by expression (4) containing no operator of definite description, insofar as only one argument satisfies our complex propositional function $S(x, z_1) \wedge P(x, y_1)$. To simplify further reasoning, let us introduce by definition the abbreviating condition that there is only one individual satisfying a certain property f. (Let us note that the functions $S(x, z_1)$ and $P(x, y_1)$ are one-place predicates, as they have only x for a variable. The function $S(x, z_1)$ may be replaced by the function $f_1(x)$, where f_1 is the predicate 'to be Supreme Commander of the Russian Army in the battle of Borodino', and the function $P(x, y_1)$ by the function $g_1(x)$, where g_1 is the predicate 'to be born in 1745'.)

The following generalised abbreviation may be introduced for this condition:

$$\exists!\, x f(x) \equiv \exists\, x f(x) \wedge \forall u\,[f(u) \rightarrow (u = x)] \tag{6}$$

(let us note that neither the right nor the left parts of expression (6) contain a context, that is, a generalised expression for the function $P(x, y_1)$, that is, the function $g(x)$).

To define expression (3) in a generalised form, let us now use the function $g(x)$, that is, our context, and expression (6). We shall obtain this expression:

$$g[(\iota x) f(x)] \equiv \exists! x f(x) \wedge g(x). \tag{7}$$

Our generalisations will thus result in a contextual definition in use for the ι-operator.

As for the syntax of the ι-operator,[1] it is simple enough when this operator is used in the proper sense, that is, when the description satisfies the conditions of existence and singularity. Definition (7) may in this case be used with regard to any function containing a description as its argument. If the ι-operator is used in the proper sense, it does not matter in what way it is eliminated on the basis of definition (7).

Supposing we have the expression:

$$\neg \{g[(\iota x) f(x)]\}. \tag{8}$$

To eliminate the ι-operator, definition (7) may be applied in two ways:

1) we may first substitute the Dfn of expression (7) for the expression in braces in (8) and add the negation sign before the braces. We shall obtain this result:

$$\neg \{\exists! x f(x) \wedge g(x)\}; \tag{9}$$

2) function g in (8) may be regarded as $\neg g$. We obtain $\neg g[(\iota x) f(x)]$. To apply definition (7) to expression (8), let us formulate the former in this way:

$$\neg g[(\iota x) f(x)] \equiv \exists! x f(x) \wedge \neg g(x). \tag{10}$$

After eliminating the ι-operator in expression (8) we shall obtain:

$$\exists! x f(x) \wedge \neg g(x). \tag{11}$$

Thus, if conditions of existence and singularity are satisfied, that is, if $f(x)$ satisfies only one single individual, the expression

$$\exists! x f(x) \cdot \neg \{\exists! x f(x) \wedge g(x)\} \sim \exists! x f(x) \wedge \neg g(x) \tag{12}$$

[1] H. Reichenbach. *Op. cit.*, p. 47.

is a tautology (the consequent of implication (12) establishes the equivalence of (9) and (11)). That means that the ι-operator as an argument of some function may be eliminated in an arbitrary order. The use of the ι-operator in the proper sense shows that it has the syntactical properties of a proper name. Matters become more complicated, however, when the ι-operator is not used in the proper sense, that is, when there is no individual or there is more than one individual satisfying the function $f(x)$. In such cases one has to introduce into the formula containing the ι-operator a symbol indicating the domain of application of that operator, that is, indicating the part of the formula to which we intend to apply definition (7). To indicate that domain, let us introduce brackets of the shape $< >$. Expression (8) will then have the following two interpretations:

$$\neg < g[(\iota x) f(x)] >, \tag{13}$$

$$< \neg g[(\iota x) f(x)] >. \tag{14}$$

In expression (14) the negation refers to the predicate g.

For greater clarity, this expression might have been written as:

$$< \bar{g}[(\iota x) f(x)] >.$$

Interpretation of expression (8) as (13) yields the meaning of expression (9), while its interpretation as (14) yields the meaning of expression (11).[1] These two expressions are nonequivalent if, for instance, the descriptum does not exist.

Let us consider an example. Supposing we have the description 'The person who has lived to be 250'. It appears as the argument of the function $g(x)$, namely, 'x takes strong alcoholic drinks'. Thus, the context is, 'The person who has lived to be 250 takes strong alcoholic drinks'. We may define

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[1] For cases where the ι-operator is not used in the proper sense, where the descriptum is absent, the requirement of singularity has no meaning any longer and expression (7) must have the form:

$$g[(\iota x) f(x)] \equiv \exists x f(x) \wedge g(x).$$

That is important by syntactical criteria as well: if the Dfn of definition (7) is negated, it becomes obscure just what is negated, existence or singularity. Russell's theory of descriptions has to be made more precise in this sense. Thus, in solving the problem of singularity, we ought to assume that the problem of existence has already been solved positively.

this expression in accordance with expression (7). These propositions may be negated in two ways, see expressions (13) and (14). By negating the proposition 'There exists a person who has lived to be 250, and he takes strong alcoholic drinks', we obtain the expressions: 'It is not true that there exists a person who has lived to be 250 and that he takes strong alcoholic drinks' (9′); 'There exists a person who has lived to be 250 and it is not true that he takes strong alcoholic drinks' (11′). Proposition (9′) is true, and proposition (11′) is false simply because the first term of the conjunction is false.

Introducing a symbol for the domain of application of the ι-operator has the advantage that sentences of the form 'That person who has lived to be 250 and who takes strong alcoholic drinks' (15) cease to be meaningless. If the domain of the application of the ι-operator is the same as in expression (9′), proposition (15) has to be regarded as false, since the negation of that which is its equivalent by definition, i.e., proposition (9′), is true. If the domain of the application of the ι-operator is the same as in expression (11′), proposition (15) is false, since that which is its equivalent by definition (i.e., proposition (11′)) is false both with and without the negation: in either case it contains a false term of conjunction. Let us note that proposition (11′) is not a negation of the sentence 'There exists a person who has lived to be 250 and he takes strong alcoholic drinks'.

Definite descriptions which have only one corresponding individual are usually contrasted in logic to indefinite descriptions which have more than one corresponding individual. Indefinite descriptions are expressed in natural language by such phrases as: 'Those x which (have a certain property)', 'Some of the objects of a class or domain (have a certain property)', 'One of those objects which...'. They are usually written in symbolic form as $\eta x f(x)$ ('those x which satisfy the function $f(x)$'). An indefinite description, just as a definite one, may be defined by means of a contextual definition in use, that is, by describing the schema of explicit definitions of those contexts which include the η-operator:

$$g[\eta(x) f(x)] \equiv \exists x [f(x) \wedge g(x)].$$

The Dfn of the definition, as distinct from expression (7), does not include the requirement of singularity. The syntax of indefinite descriptions is more complicated than that of

definite descriptions. It is important to note that in defining the meaning of the function in symbolic logic indefinite descriptions are not used as its arguments.

The sphere of application of contextual definitions in science and in everyday life is extremely extensive. In our view, it is not by chance that Reichenbach argues the existence of only two basic classes of definitions differing in their structure and purpose: definitions *per genus proximum et differentiam specificam* and definitions in use.

So far we have been considering only those contextual definitions in use in which the Dfd is a certain proposition (context) whose principal terms are defined simultaneously. The concept of contextual definitions in use may apparently be generalised in such a way that the set defined by this concept should include definitions of object functions considered in the context of variables.

Let us suppose that we have to define a trigonometrical sin-function. The definition of this abstract object is reduced to the definition of the context sin x, an attempt being made to give a definition of this function which would ensure the most effective mode of defining its values. The function sin x for all values of x may be defined in different ways. It may for example be defined by the series into which sin x may be expanded (it is the sin x, and not sin that is expanded into a series). The sin-function may also be defined by a table of the values of sin x, combined with the rules for compiling it. It may be said that *the sin abstraction is a certain function whose value is described as the values of the function sin x together with the rules for their calculation.*

In other words, to define the function sin, we reduce its definition to defining the context of sin x. This ensures the effectiveness of the definitions created and allows us to eliminate the context defined, reducing it in every concrete case to the appropriate values (numbers). We may then define again not the sin x, but the abstract object sin. We use here the abstraction operator λ, which constructs expressions $\lambda x \sin x$ out of sin x where the variable x is already bound. This abstract object $\lambda x \sin x$ may again be eliminated and reduced to the function sin x together with the free variable through the operation of *application*: $(\lambda x \sin x)y = \sin y$ (through the operation of application of $\lambda x \sin x$ to y we obtain sin y).

In a similar way, definitions of abstractions formed through

isolating abstraction are usually reduced to definitions of corresponding propositional functions. Thus, instead of defining causality or equality, we define the propositional functions '*a* is the cause of *b*', '$x=y$'. Let us note, however, that in the examples of defining abstractions cited here we reduce their definitions to defining contexts which are regarded as ordinary, normal definitions (see 3.2).

1.7. CLASSIFICATORY AND GENETIC DEFINITIONS

There is a distinction made in logic between classificatory and genetic definitions. The former include definitions *per genus proximum et differentiam specificam* which distinguish the Dfd from the objects of a certain domain (the latter being explicitly indicated in the definition—the *genus*) through describing the specific features of the Dfd fixed in the Dfn (the *differentia specifica*).

Such definitions may be illustrated by the following examples: 'The rhombus is a flat quadrangle in which all sides are equal', 'Man is an animal capable of labour, articulate speech and thought'. In the first definition the set of flat quadrangles is the genus, and the property of 'having all sides equal' is the specific distinctive feature. This feature is used to single out that subset of the set of flat quadrangles whose elements are rhombuses.

Forming definitions through genus and the distinctive features assumes objects pre-existing definition, as well as the existence of names for the Dfd in a certain language. It is also assumed that the Dfd may be specified by the properties—'the distinctive features'. To put it differently, what is meant is the fact that these definitions are of *analytic* nature in the sense of 1.3. They are usually formulated as real definitions. Definitions through genus and the distinctive features thus serve to delimitate objects (resp. the meanings of terms) that have already been singled out in the experience of mankind, those in which elementary systematic classificatory relations have been established. These definitions were introduced into logic already by Aristotle. Later on, however, the tendency developed in logic to extend this class of definitions beyond its proper bounds, sometimes unjustifiably: various types of the properly synthetic definitions came to be included in this class. The concept of distinc-

tive feature was also extended. Distinctive features came to be interpreted as any feature specifying the Dfd, not only the properties of the object defined. This class of definitions was extended along other dimensions as well, on the assumption of the possibility of exhaustive logical analysis of definition taken out of context, as an isolated entity, and not as an element or component of a scientific discipline as a whole.

Supposing that analysis of reality results in a description of a certain new object through its characteristics. We may establish a correspondence between this object and a new name that has not previously existed in the former language *S*. This new name is introduced through a nominal semantic definition. (Cf. 'That which has such and such characteristics will be named in such and such a way'. For example, 'That substance which is secreted by the Langerghans islands of the pancreas will be called insulin'.) As it stands, this definition is not a definition *per genus proximum et differentiam specificam*. Later on, however, it may be transformed into a real analytic definition interpretable as a definition *per genus proximum et differentiam specificam*. For instance, the definition 'Insulin is a secretion of the Langerghans islands of the pancreas' is already such definition: it specifies an object (insulin) through the genus and the distinctive feature.

This sort of translation and reduction is done rather frequently, which creates the illusion that definitions *per genus proximum et differentiam specificam* are a very large class of definitions. The possibility of reductions of this sort seems to have been the reason why definition rules in manuals on logic have almost invariably been formulated with regard to definitions *per genus proximum et differentiam specificam*, which are analytic and real (cf. the formulation of the first rule as that of *equal extent* of the Dfd and the Dfn). Apparently the tendency itself for reductions of this sort was conditioned by the genus-to-species systematisation of experience which seems the most simple and therefore most natural.

Of the definitions *per genus proximum et differentiam specificam* the so-called genetic definitions were first singled out in the history of logic. Definitions *per genus proximum et differentiam specificam* were used to single out a certain subset of objects in a certain set of existing objects (the genus)

by certain *properties*, while genetic definitions specify the objects defined through descriptions of *the modes of their formation, emergence, or construction.*

Manuals on formal logic usually establish a correlation between definitions *per genus proximum et differentiam specificam* and genetic definitions.

Thus, a correspondence is established, for instance, between the genetic definition of the circle 'The circle is a closed curve formed by rotating segment AB of a straight line around point A on a plane; point B will then describe a curve called a circle' and the definition *per genus proximum et differentiam specificam*: 'The circle is a closed flat curve in which all points are equidistant from one point (the centre of the circle) lying on the same plane as the curve.'

However, some manuals on traditional logic stress that many genetic definitions are not merely more natural but also more effective than definitions *per genus proximum et differentiam specificam*, since they describe the procedure ('a recipe') for producing a certain object (cf. '*Borsch* is cabbage soup', 'Sauerkraut soup is soup made of sauerkraut', 'Sorrel soup is soup made of sorrel', which are definitions *per genus proximum et differentiam specificam*, and recipes in cookbooks describing the procedures for cooking *borsch*, sauerkraut soup, sorrel soup, i.e., genetic definitions). The same kind of situations arise in medicine or industrial technology.

Genetic definitions have independent significance in science, they are not modifications of definitions *per genus proximum et differentiam specificam* equivalent in all respects to the latter (these were sometimes characterised in the history of logic as definitions through the description of the essence of the object, since the specific feature proved to be the essential feature). Thus, genetic definitions played a different role in the development of knowledge, distinct from the function of classificatory definitions.

Genetic (or, to use another term, inductive) definitions are frequently used in science when the corresponding definition *per genus proximum et differentiam specificam* has not yet been formed. In chemistry, when a certain substance proves to be, in a sense, an unopened 'black box', it is usually defined through the description of the methods and conditions for obtaining it from other substances. The elaboration of the definition of the essence in such cases does not cancel

the corresponding genetic definition, as the latter may prove to be more effective for the solution of many problems. Thus, any individual natural number may be given by a genetic (inductive) definition (see 3.4). Such definitions are not temporary modifications of definitions *per genus proximum et differentiam specificam*, as they are used as part of genetically constructed theories; the method of constructing such theories requires the use of definitions of this type and not of some other. Genetic definitions are also widely used in empirical sciences: they may be used, for instance, in introducing idealised objects into science.[1]

1.8. DEFINITIONS THROUGH ABSTRACTION

The so-called definitions through abstraction play an important role in science.

They are *definitions involving the positing of certain sets or corresponding properties as 'abstract objects' through establishing equality-type relations between the objects being studied and introducing certain names for them.* They represent a stage in the formation of the corresponding concepts.

Equality-type relations are defined axiomatically, usually as follows: the equality-type relation R is a binary relation between the objects x and y in the domain D, which satisfies the following conditions (or axioms):

(1) the relation R is reflexive in the domain D: xRx;

(2) the relation R is symmetrical in the domain D:

$xRy \longrightarrow yRx$;

(3) the relation R is transitive in the domain D:

$xRy \wedge yRz \longrightarrow xRz$.

This definition of the relation R covers various concrete relations defined for the corresponding domains of objects. Of this nature is the equality relation itself ($=$), as it is reflexive (each object x of the domain under analysis equals itself: $x=x$); symmetrical (it is true for any pair of objects x and y of the domain under study that if $x=y$, then $y=x$); transitive, since, if $x=y$, and $y=z$, then $x=z$.

[1] See D. P. Gorsky. *Voprosy abstraktsii i obrazovaniye ponyatii* (Problems of Abstraction and the Formation of Concepts). Akademiya Nauk SSSR Publishers, Moscow, 1961, pp. 276-85.

This definition also holds for such relations as: 1:1-correspondence (of sets), equilibrium (of bodies), interchangeability (of commodities in the market), simultaneity (of events), similarity (of figures), parallelism (of straight lines on a plane), etc. For this reason the relations listed here will be included in the class of equality-type relations. As for the relations of the type 'greater' or 'smaller', they are not equality-type relations, as they do not satisfy axioms (1) and (2); for instance, it is not true that $x>x$ (it is not true that each object of the domain is greater than itself), it is not true that if $x>y$, then $y>x$.

This equality type relation R is often called an equivalence relation. In studying the correlations of objects of a certain domain D, for which an equivalence relation R is defined, we shall use a certain principle called the *principle of abstraction*. We shall formulate this principle as follows: the objects of the domain D with an equivalence relation R defined on them will constitute a certain set M_1 with the property P_1 common to all its elements, if and only if all the objects of the domain are equivalent in pairs.

Assume that we have at our disposal axioms (1)-(3) and the condition $\forall x \exists y(xRy)$: 'For any object of the domain there exists another object that stands in relation R to it.' Our domain D will include the objects $O_1, O_2, O_3, \ldots, O_n$. We shall prove on this basis that all the objects of the domain D are equivalent in pairs (theorem 1).

Eliminating the universal and existential quantifiers in the formula $\forall x \exists y(xRy)$, we may obtain O_jRO_1, where $j \geqslant 1$ (or O_iRO_1 which is the same as O_jRO_1).

Thus we have:

O_jRO_1 (1), O_iRO_1 (2), axioms (1)-(13).

We must prove that O_jRO_i.

1. From axiom (2) and O_iRO_1 we obtain (by substituting O_i for x and O_1 for y in the axiom and applying the *modus ponens*) O_1RO_i (3).

2. From O_jRO_1 and O_1RO_i we obtain: $O_jRO_1 \wedge O_1RO_i$ (BK:1,3).

3. From axiom (3) with the necessary substitution and from $O_jRO_1 \wedge O_1RO_i$ we obtain: O_jRO_i *(q.e.d.)*.

That means that under conditions formulated above the objects of our domain are equivalent in pairs, that is to say, the correlation $\forall x \exists y(xRy)$ obtains. That means that all

the objects of the domain D will be identified with one another through relation R and form the set M_1.

Using theorem (1) proved here as the basis and applying the principle of abstraction to it, we may prove theorem (2), namely, that all elements of the set M_1 will have the property P_1.

Let us now prove that an equality (equivalence)-type relation divides the domain D into classes $M_1, M_2, M_3, \ldots, M_k$ having no common elements, inasmuch as there will be objects in the domain D that do not stand in relation R to the objects $O_1, O_2, \ldots, O_n$ (the multiplicative principle formulated by Russell).

Let us stipulate that the domain D will include the object O_1^1 which does not stand in the relation R to any of the objects $O_1, O_2, O_3, \ldots, O_n$. It will not then be included in class M_1. But if there are other objects $O_2^1, O_3^1, \ldots, O_n^1$ which stand in the relation R to O_1^1, they will form class M_2—in accordance with the theorem proved above. None of the elements of class M_2 will be included in M_1 since they became glued together with the element O_1^1, which does not stand in the relation R to any of the elements $O_1, O_2, O_3, \ldots, O_n$. These classes cannot have any common elements. All that has been said above may be repeated with reference to all the objects of the domain D that are not connected by the relation R capable of dividing the domain D into the sets $M_1, M_2, M_3, \ldots, M_k$ that have no common elements. For each of the properties $P_1, P_2, P_3, \ldots, P_k$, in which the elements of each of the sets $M_1, M_2, M_3, \ldots, M_k$ prove to be indistinguishable, one may introduce a special name.

To explain what has been said here we shall consider concrete examples. Let the domain D consist of diverse commodities, only some of which (e.g., $O_1, O_2, O_3, \ldots, O_n$) stand in the relation R to each other—the relation of *direct mutual exchangeability* (on the market). These commodities are thus equated with each other in the process of exchange. They prove to be equal to each other in certain respects, this equality making them indistinguishable from each other, uniting them in one class M_1. That which is common to all these directly exchanged commodities may be called the value of these same commodities (not value in general). R is used to reveal the common property of the given set M_1 of these commodities of equal value.

There may be most diverse commodities O_1^1, O_2^1, O_3^1, ... , O_n^1 in the same domain such that they are also directly mutually exchangeable, but none of them may be exchanged for any object O_j. With the aid of R we may determine a certain property P common to them (namely, 'to have such and such value') which will permit to include all the commodities O_j^1 in the set M_2 having no common elements with the set M_1, etc.

Let the domain D contain a set of various assemblages of objects; for the elements of some of these assemblages a relation R is established—the relation of *1:1-correspondence* (e.g., we establish a 1:1-correspondence between the fingers of the right hand, the petals of the buttercup, the number of the apices of the pentagon, the number of letters in the word 'seven', etc.; all of them prove to be equal in number). These assemblages become equated with each other—they become indistinguishable. The common feature of the assemblages standing in a 1:1-correspondence to each other is their power, the cardinal number of these assemblages, which may be named, for example, by the term 'five'. Here again we have singled out the property characterising these assemblages rather than the properties 'to have power' or 'number' in general.

In a similar way the relationship between bodies in equilibrium cannot yield the property of weight in general but only the property of weight for each of the bodies in equilibrium.

Let us now assume that for the elements of a certain set the relation is established 'to be identical with regard to a certain directly (visibly) observable property P' (an equality-type relation R). As distinct from other relations considered above, this one includes a description of some *directly observable* property P. The set of objects identical with regard to P is composed in the following manne: we choose milk, for instance, and then select those objects that are identical with it in regard to a certain property (colour): snow, swans, cherry blossom, lime, etc.

As the common property of all these objects has to do with their colour and we may conclude directly that they are of identical colour, we may single out that common property that exists in all of them, namely, the property of some definite colour, and give it a name—e.g., white. In this

case the property of colour in general has not yet been singled out.[1]

So we see that the procedure described above and satisfying certain conditions (namely, the fact that any object of some set within the domain is equated with some other object through the relation R) cannot yet yield the properties of number in general, value in general, weight in general, colour in general so that corresponding concepts might be formed. We can merely single out such properties as 'to exist in such and such numbers' (to have definite power), 'to have definite value', 'to have definite weight', 'to have definite colour'.

To form abstractions of a higher order, we have to go on with the analysis. Let our domain D contain the sets M_1, M_2, M_3, ... , M_k having no common elements and formed in the manner described above. Let these sets consist of the following elements:

$M_1 \{O_1^1, O_2^1, O_3^1, \ldots, O_n^1\}$,
$M_2 \{O_1^2, O_2^2, O_3^2, \ldots, O_n^2\}$,
$M_3 \{O_1^3, O_2^3, O_3^3, \ldots, O_n^3\}$,
.
.
.
.
$M_k \{O_1^k, O_2^k, O_3^k, \ldots, O_n^k\}$.

In each of the sets M_1, M_2, M_3, ... , M_k the elements are 'glued together' due to the fact that an equality-type relation R obtains between them. However, arbitrary elements of each of the different sets M_1, M_2, M_3, ... , M_k do not stand in relation R to each other. In other words, the relation R is established between the elements $\{O_1^1, O_2^1, O_3^1, \ldots, O_n^1\}$, that is, between the elements O_i^1, between the elements $\{O_1^2, O_2^2, O_3^2, \ldots, O_n^2\}$, that is, between the elements O_i^2, etc. But $O_i^1, O_i^2, O_i^3, \ldots, O_i^k$ do not form pairs with regard to the relation R.

[1] General properties of objects corresponding to the so-called *dispositional* predicates ('to be soluble', 'to be resilient', 'to conduct electricity', etc.) are singled out in a way different from the above. This seems to be explained by the fact that in discovering these properties we proceed from a physical *experiment* revealing this property in various individuals.

That is precisely what permits the transition to abstractions of a higher level (number in general, value in general, colour in general, weight in general, etc.).

Thus, the first stage of the procedure described above enables us to single out, by means of definitions through abstraction, such abstractions as the number 'five', the number 'ten', the number 'twenty', etc. Strictly speaking, we cannot yet use the word 'number' in our language, because the corresponding abstraction has not yet been formed. The words 'five', 'ten', 'twenty' were introduced respectively as names of a set of assemblages equal in number, e.g., the number of fingers of one hand, or a set of assemblages equal in number to the fingers of both hands, etc. That means that our domain D will in this case contain the sets M_1, M_2, M_3, ... , M_k forming no intersecting pairs, that is, having no common elements.

It may be shown that if the sets M_1, M_2, M_3, ... , M_k had at least one common element they would coincide with each other, that is, become indistinguishable. Our domain D may thus be regarded as a set including the sets M_1, M_2, M_3, ... , M_k (as well as the corresponding properties serving to identify their elements) as its elements.

All the sets in the example concerning the formation of numbers differ in their individual power corresponding to the various numbers denoted in our example by the words 'five', 'ten', 'twenty', etc.

We can now ignore the individual characteristics associated with the difference in the power of the sets M_1, M_2, M_3, ... , M_k corresponding to the various numbers, singling out their *common* feature (that is, we can perform the identification abstraction). The common feature of the sets of different power characterised by the different numbers (which is manifested in the difference between their names) will apparently be their *power in general* or *number* (cardinal number) *in general.*

In a similar way, the common feature of the classes of objects having the same colour as milk, blood, etc. (for which such names as 'white', 'red', etc., are introduced) will apparently be that they have *colour in general.*

Number in general is thus a set of sets M_1, M_2, M_3, ... , M_k where each set M_j is a set of sets having as many elements as there are fingers on one hand, on two hands, etc. The fingers of the hand, of two hands, etc., also form sets.

What is the abstraction level of 'number in general'? Material objects (e.g., man's fingers and toes and other material objects of which standard sets consist) will be regarded as *zero-level* objects. Sets corresponding to standard sets of some kind will be viewed as *first-level* objects. Sets of sets consisting of the same number of elements as the standard sets will be *second-level* abstraction objects (these are our sets M_1, M_2, M_3, ... , M_k). The set of sets M_1, M_2, M_3, ... , M_k (that is, the set of sets all of which have the same number of elements) will be *third-level* abstraction objects. All these sets register certain aspects of reality abstracted from it step by step.

It is very difficult to express the *extensional definition* (see 1.10) of number in general in ordinary language. It may, for instance, be of this form: number (cardinal number) is a set of sets (M_1, M_2, M_3, ... , M_k), each of which is also a set of sets numerically equal to some definite set (e.g., the number of fingers of one hand). Or, number (cardinal number) is that common feature which we find in the set of sets all of which are numerically equal to each other.

In a similar way, colour is that common feature which all sets of identically coloured objects possess, or colour is the set of all sets of identically coloured objects. It is easy to see that 'colour' is a second-level abstraction.

In the examples considered here the sets M_1, M_2, M_3, ... , M_k have been formed by *definitions through abstraction.*

Later on, however, we made use of *identification abstraction* to form abstractions of a higher level: in forming the set of sets M_1, M_2, M_3, ... , M_k we identified the sets with each other, or 'glued them together' (at the same time ignoring the differences between M_1, M_2, M_3, ... , M_k) constructing a new set D on the basis of some property common to them all; we said that that property was the common feature existing in the sets M_1, M_2, M_3, ... , M_k, and named that property by some term or other: 'number', 'value', etc.

Naturally, definitions through abstraction may be used when conditions have been prepared for equating objects through an equality-type relation R while the property which serves as the basis for equating them is either undiscovered or has no name yet. Thus, in establishing a 1:1-correspondence between sets we revealed their property of 'having identical power', 'one and the same numerical characteristic', giving the sets of numerically equal, equipotential

objects and the corresponding general properties the names 'five', 'ten', etc. This process is based on man's practical ability to match various collections of objects, without any knowledge of counting or conception of natural number. In distinguishing the various colours (it being assumed that the names for the colours have not yet been introduced) we formed sets of identically coloured objects through partial identity relation (an equality-type relation R), proceeding from our ability to identify and distinguish between objects through our sense organs from some directly observable properties.

The very fact that there are names for the relevant properties of objects in the language is proof enough that these properties have been singled out in human experience, the question now being that of passing on this experience to the new generations. For this purpose, in the case of directly observable properties, we turn to the so-called *ostensive definitions* (see 1.11). These definitions permit to perform the identification abstraction: proceeding from the identity and difference in the use of names in speech, the subject singles out something that is common to all the objects, neglecting their differences. The singling out of sensually perceived properties on the basis of language is simplified, and so is the mastering of operations performed on the abstractions (including numbers and operations on them). This abstraction will be called *identification abstraction based on language.* It should be distinguished from a type of abstraction known as definition through abstraction, which is also used to identify objects with each other through an equality-type relation. Finally, we made use of still another identification abstraction disregarding the differences between the sets M_1, M_2, M_3, ... , M_k.

In this case we compare different variants of one and the same property. This seems to be a special kind of identification abstraction. The last two types of identification abstraction are used in the historical development of cognition when mankind acquires fundamentally new experiences, particularly when new conceptions are formed.

When an equality-type relation is established between rigid discrete abstract objects like straight lines on a plane, geometrical figures, etc., definitions through abstraction are applied for identifying a new property in the objects under study according to the scheme outlined above (see

pp. 62-66). We can, for instance, determine the property of equidirectionality through the relation of parallelism, which is an equality-type relation. Thus, by selecting straight lines parallel with the given straight line AB, we form the set M_1 of straight lines parallel with AB. By selecting straight lines parallel with A_1B_1 (no relation of parallelism existing between AB and A_1B_1), we shall form a new class of parallel lines M_2, etc. By applying abstraction to the straight lines parallel with AB, we can single out the property 'to have the same direction as the straight line AB'. In a similar way, by analysing class M_2, we can determine the property 'to have the same direction as the straight line A_1B_1'. That general property which exists in these sets (that is, in the sets of equidirectional straight lines) will be *direction of a straight line in general.*

The relation of similarity may be used to form the various classes of equisimilar triangles, the various classes of equisimilar quadrangles, etc. This may serve as the basis for forming the concept of the geometrical figure in general.

In forming the concept of value through the relation of exchangeability of commodities, we resort to a number of restrictions and idealisation assumptions, as the establishment of equality-type relations between the objects is preceded by their measurement and counting, while the objects themselves are now clearly discrete, now practically continuous.

Why should the procedure described above be counted among definitions?

The point is that definitions through abstraction are used to single out or discover a certain *new property* in the objects for which an equality-type relation exists. This property (and the corresponding class) are correlated with a certain name. At the initial stage in the study of familiar objects, the meaning of the name introduced for them is precisely the description of the procedure for discovering a certain new property and forming the corresponding set constructed by means of establishing an equality-type relation between the objects under analysis. The result of this procedure is formulated as an *explicit* definition.

The formation of sets (and the corresponding properties) of the type M_1, M_2, M_3, ... , M_k involved here is conditioned on applying to the objects under study the kind of identification abstraction that is realised through establishing

an equality-type relation R between them. By correlating each of the sets M_1, M_2, M_3, ... , M_k with a special name, *we form certain propositions that are definitions* (through abstraction). Constructing abstractions of a higher level and forming definitions for these abstractions are already based on application of a special type of identification abstraction to objects representing the sets M_1, M_2, M_3, ... , M_k. In introducing new names for the set of sets M_1, M_2, M_3, ... , M_k ('number', 'value', 'figure', etc.), we also form *explicit definitions.*

Can definitions of these abstractions of a higher order be also regarded as definitions through abstraction? It seems that this question will have to be answered in the affirmative, since the formation of such abstractions as number, value, figure, direction, colour, etc., necessarily goes through the stage of forming and defining abstractions of a lower level (i.e., sets M_1, M_2, M_3, ... , M_k), constructed by means of procedures that are definitions through abstraction in 'pure form'.

Do these definitions satisfy the requirement of nonambiguity, does R single out only one Dfd for which a new name is introduced? Couldn't we, for instance, use the relation of 1:1-correspondence to single out not only the property 'to be number x', but also such properties, common to all sets, as 'to be rigid objects', 'to be sensually perceived objects', etc.? Isn't the relation of exchangeability of commodities used to single out other properties (e.g., the property 'to satisfy man's needs') than 'to have value'?

Let us give only a few reasons to prove that these definitions are used to single out only one new property unperceived by the senses and that, consequently, these definitions satisfy the requirement of nonambiguity, of interchangeability of the Dfd and the Dfn.

First, to carry out definition through abstraction, we use an equality-type relation to discard the numerous properties of sensually perceived objects equated with each other by the equality-type relation R. We begin to regard them as certain objects without qualities that are characterised merely by the fact that an equality-type relation obtains between them.

In defining number we proceed from the abstract concept of set, in defining value, from the concept of commodity (all the objects are characterised only with regard to this pro-

perty common to them all). So the question can only be formulated like this: do these definitions single out just one new property unperceived by the senses or are there many such properties?

Second, an equality-type relation is external and cannot therefore create any new properties in the abstract objects—the classes between which it is established: it can only reveal the properties already in existence.

Third, this newly discovered property may only be 'kindred' in nature with the relation R: it must ensure the realisability of this kind of relation between objects.

Fourth, if there were several such new 'external' properties 'unperceived by the senses' and 'kindred' with the relation R, they would indeed be indistinguishable. The relation R would be incapable of individualising them, although they might be individualised by other means. Such cases, however, are unknown to science.

The above might be repeated, *mutatis mutandis*, with reference to singling out sensually perceived properties on the basis of definitions through abstraction. Identification of objects with regard to certain properties (that is an equality-type relation) may serve as the basis for singling out the property 'to be red' (that is the common property of all objects having the same colour as blood), 'to be white' (that is the common property of all objects having the same colour as milk) and the corresponding sets of white and red objects. In this way we disengage ourselves from all the other properties common to all red objects and all white objects (e.g., the property of 'having a smell', 'having a form'). In establishing the common feature of red objects and white objects which are regarded from the point of view of colour only, we may single out *only one* property, that of 'having colour'.

It should be mentioned that the definition of natural quantitative numbers according to Frege and Russell given above appears debatable to many mathematicians. It is debatable if only because it explains intuitively much clearer things through intuitively less clear concepts. Apart from this, numbers, which appear in mathematics as signs, that is, in their sensually perceptible form, are reduced to abstractions of an extremely high level in definitions through abstraction.

The procedure of generating the natural series of numbers

from an empty set, based on the idea of similarity of classes (or sets), seems rather artificial. Let us describe this procedure. We have shown that two similar classes (those for which a 1:1-correspondence holds) have the same cardinal number. Proceeding from the empty set concept $\wedge$ we can now define the number '0' as the cardinal number of the class $\wedge$, i.e., the class of all nonexistent things. The empty class is defined as a class without elements or as a class of all elements that are not identical with themselves (it being assumed that such objects do not exist and that we are dealing with objects to which constructivisation and abstraction have already been applied). Further, the number '1' is defined as the cardinal number of the class whose only element is the empty class. The number '2' is defined as the cardinal number of the class whose elements are the empty class and the class whose only element is the empty class, etc. For the sake of clarity the series of numbers generated from the empty class of numbers may be represented as: 0, {0}, {0, {0}}... , etc.

It should be pointed out that Russell himself, in his *An Introduction to Metamathematical Philosophy*, refers rather ironically to this mode of generation of numbers, saying that it is very much like producing a crane from an empty hat.

Constructivist mathematics rejects the Frege-Russell definition of numbers, defining them instead purely syntactically as signs, as series of strokes following each other. They may be distinguished by the length of the series, operation rules applicable to them, etc.

There has been a discussion among logicians of the axioms underlying definition through abstraction. For instance, only two axioms were sometimes included among those characterising the relation R, the axioms of symmetry and transitivity of the relation R; as for the reflexivity of this relation, an attempt was made to deduce it from the other two, which resulted in an error.[1] H. Weyl introduces the requirement of reflexivity of the relation R among the axioms.

W. Dubislav, just as most mathematicians, adds the requirement that the domain of the relation R be nonempty

[1] See S. A. Yanovskaya. *Metodologicheskiye problemy nauki* (Methodological Problems of Science). Mysl Publishers Moscow, 1972, pp.55-58.

to the axioms of symmetry and transitivity and deduces, on this condition, the property of reflexivity for R. S. A. Yanovskaya believes this approach to be methodologically sound as it is linked with the characteristics of the domain of the relation itself where objects pre-exist definition, so that one may regard the common property abstracted from the objects as being singled out by the definition, not created by it. Yanovskaya reproduces here Dubislav's proof, although the latter contains an error, as she herself pointed out in her lectures on more than one occasion.

Indeed, if there is an object a in the domain of the relation R such that it stands in the relation R to some other object (that is, the axiom $\exists x \exists y (xRy)$ obtains), and it is also known that R is symmetrical and transitive, the axioms

(1) $\exists x \exists y (xRy)$,

(2) $\forall x \forall y (xRy \rightarrow yRx)$,

(3) $\forall x \forall y \forall z (xRy \wedge yRz \rightarrow xRz)$

cannot yield the property of reflexivity $R : xRx$. Indeed, by eliminating the quantifiers in (1), we shall obtain:

aRb, where a and b are certain individual objects. (4)

By eliminating the quantifiers from (2), we shall obtain:

$aRb \rightarrow bRa.$ (5)

By applying the rule of separation, we shall obtain from (5) and (4):

$bRa.$ (6)

By applying the rule for introducing conjunctions to (4) and (6), we shall obtain:

$aRb \wedge bRa.$ (7)

By eliminating the quantifiers in (3) and on the assumption that $x=a$, $y=b$, $z=a$, we shall obtain:

$aRb \wedge bRa \rightarrow aRa.$ (8)

By applying the separation rule, we obtain from (8) and (7):

$aRa.$ (9)

It is impossible, however, to introduce the universal quantifier for (9), as the rule of eliminating the existential quantifier was used in obtaining it.

It is easy to show that, if one takes the axiom $\forall x \exists y (xRy)$ instead of axiom (1), that is, if for any x there exists an y such that x stands in the relation R to it, then reflexivity of R proves to be deducible from the axioms. The proof is similar to the above. The universal quantifier may be introduced into the formula aRa deduced above, so that we obtain $\forall x (xRx)$.

As the property of reflexivity of $R - \forall x (xRx)$ may be obtained from axioms (1)-(3) and the formula $\forall x \exists y (xRy)$, it is not by accident that we did not use the property of reflexivity in proving theorem (1): the formula $\forall x \exists y (xRy)$ is at least just as strong as the formula $\forall x (xRx)$.

1.9. NONPREDICATIVE AND PREDICATIVE DEFINITIONS

So-called nonpredicative and predicative definitions are distinguished in modern logic and mathematics.

Nonpredicative definitions in the special sense are those which *introduce* certain new objects through sets, the latter at the same time including the new objects introduced by the definition as their elements.[1]

These definitions should be regarded as containing a *vicious circle* as the Dfd is here defined through the Dfn which by definition includes the Dfd. To eliminate the vicious circle, these definitions are given a special interpretation; some additional assumptions are formulated for them.

The problem of predicativity or nonpredicativity usually arises in connection with real definitions.

The definition of the upper limit of the set of real numbers is often cited as a nonpredicative definition: 'The upper limit of the set of real numbers is the greatest number in this set, that is, that number which is greater than any number of this set.'

The Dfd, 'the upper limit of the set of real numbers', is here defined through the set of numbers including the Dfd

[1] Hao Wang. 'Ordinal Members and Predicative Set Theory'. *Zeiteschrift für mathematische Logik und Grundlagen der Mathematik*, Bd. 5, H. 3/4.

as its element: that which is introduced by the definition is defined through a set which includes the object introduced by the definition. Assuming that the given object ('the upper limit') did not pre-exist definition as an element of the set of real numbers (the set of real numbers is not assumed to have existed before the definition 'ready-made', as it were) we shall be dealing here with a definition containing a *vicious circle*.

If the assumption is made, however, that the set of real numbers exists in some form, is somehow 'given' *independently of our definition*, no circularity results. One does not introduce a new object in this case but merely *selects an already existing object* from a previously defined set of objects. This selection is done by pointing out the specific characteristics of the object defined. In classical mathematics such definitions are viewed as definitions in their own right. The vicious circle is here eliminated by the following assumptions: (1) The set through which the Dfd is defined is taken to be somehow given, defined and existing beforehand. (2) The definition does not introduce here a new object but singles out an already existing object from a certain whole or set. (2) is a corollary of (1).

The set of real numbers is exemplified in classical mathematics by the set of points on a straight line and is believed in this sense to pre-exist the definition. In constructivist mathematics, however, such definitions are excluded.

Another type of *nonpredicative definitions* are definitions containing a circle (not a vicious one), of which it is known in advance that the Dfd and the set which contains it as an element and through which it is defined, are defined, 'introduced' into the science, or somehow 'given' before definition. In cases of this sort it is known beforehand that the definition does not introduce a new object but merely singles out or specifies an existing object (element) of some existing set.

A fairly large class of definitions has these properties. In definitions *per genus proximum et differentiam specificam* (see 1.7) the Dfd is included in a certain set as its element and then singled out from this set by some specific feature. Thus, in defining Mars as a planet of the solar system fourth in the order of distance from the Sun, we include the Dfd, i.e., the planet Mars, in the Dfn as its component part, one of the planets of the solar system, and then distinguish it by its specific feature among other planets.

Explicit descriptive definitions (those with the ι-operator, that is, the 'that ... which' operator) also include the Dfd in the Dfn, the former then being distinguished by some characteristics from the other elements of the set (see 1.5). Of this nature are definitions like: 'This man is the tallest person among the workers of our office', 'The number "2" is that number which is greater than zero and which, being added to itself, yields its square'.

The Dfd in these definitions ('this man', 'the number "2"') is included in the set of which it is an element ('the workers of our office', 'numbers'), and then it is distinguished from the other elements by some relation to them or some operations performed on them: 'this man' proves to be the tallest person among the workers of our office, the number '2', the number which, being added to itself, yields its square (and does not equal zero).

These definitions meet all the requirements which scientific definitions should satisfy thanks to the assumption that all the elements of the set or domain that include the object defined somehow pre-exist definition, having been defined in advance.

It is precisely the nonpredicative definitions in the special sense to which Bertrand Russell's *reducibility axiom* applies; it says that for objects defined in this way *there must be other means of introducing sets including the objects defined as their elements*, independent of the nonpredicative definition; *the nonpredicative definition does not contain in this case a vicious circle*. That is what makes nonpredicative definitions usable. In the framework of classical (Cantor's) mathematics we may accept the definition of the upper limit of the set of real numbers, as this set (the Dfd of the definition included) may be exemplified (and thereby in a certain sense given) by the set of actually existing points on a segment of a straight line (0, 1).

We resort to definitions *per genus proximum et differentiam specificam* to define objects and sets of objects of material reality without limitation precisely because the Dfd and the Dfn exist in material reality before any definition. Moreover, we imply that the Dfd and the Dfn may be defined exhaustively by ostensive definition or list. The introduction of a set through the properties of the elements does not form or create the elements of this set or the collection itself of such objects but distinguishes that which exists independent-

ly of definition. It is true, however, that we perform a significant act of abstraction in transforming really existing objects into elements of a set that are indistinguishable from the point of view of a certain property P, and collections of objects into 'abstract objects' (sets, classes) in which elements are 'glued together'. Thus, one should distinguish between definitions containing a vicious circle (definitions for which interpretations are thought up to avoid circularity) and definitions containing a circle that is not vicious. There are also definitions containing no circle whatever. These definitions are called *predicative*.

Typical predicative definitions are represented by any inductive definition, any ostensive definition, and those nominal definitions which are viewed as abbreviations for specific properties of a certain class of objects.

1.10. EXTENSIONAL AND INTENSIONAL DEFINITIONS

Definitions (nominal definitions first and foremost) are sometimes subdivided into extensional and intensional [1] depending on whether the meaning of the term is defined through its extensional or intensional.

In natural language, when the class is finite and comparatively easily registered, the term denoting any element of this class may be defined not only through a description of the corresponding properties *(an intensional definition)* but also through listing the elements denoted by this term *(an extensional definition)*. Supposing we want to define the meaning of the term 'continents'. We can define it intensionally: 'Large areas of land surrounded on all sides or almost all sides by the waters of the World Ocean are called continents.' An extensional definition would look like this: 'The term "continents" is used to denote only Europe and Asia (one continent), Africa, South America, North America, Australia, and the Antarctic.'

If the class is very numerous (to say nothing of practically infinite classes), an extensional definition of the corresponding term cannot be realised: one cannot do more than cite a few examples to clarify the corresponding term.

[1] See H. S. Leonard. *Op. cit.*, pp. 289-90.

We often define various mathematical, as well as logical, functions by means of tables. In such cases the Dfn is not merely a table for the values of the Dfd, but the table plus the rules for its construction, the rules for calculating its values (cf. the tables defining the trigonometric functions). Sets of objects are sometimes defined in mathematics by the rules of generating their elements from a set of elementary units. In this case one can hardly speak of purely extensional definitions or include the rules and generation procedures themselves among the extensional characteristics of the corresponding term. Equally, they cannot be included, it seems, among the intensional characteristics of the object (and the corresponding term). The point is that operations (such as '+', 'following x', etc.) are not characteristics of those objects (and corresponding terms) to which they are applied in concrete cases and are in this sense essentially different from such characteristics of objects as their properties and relations (resp. from the intensional characteristics of concepts and meanings of terms).

1.11. OSTENSIVE AND VERBAL DEFINITIONS

A distinction is frequently made between ostensive and verbal difinitions. Verbal definitions are formulated on the basis of sign activity in the broadest sense as well as the information contained in the signs. For formulating ostensive definitions, sign activity alone is not sufficient: one has to go beyond its limits and turn to direct experiential activity, including practical activity.[1]

All that is needed for comprehending definitions of meanings of words in natural language as registered in dictionaries as well as definitions constructed in the sciences and scientific theories is a knowledge of the language in which the experience of mankind is registered. It is from language that the lexicologist and the theoretician of science proceed in formulating definitions (these are verbal definitions). Definitions formulated at the level of empirical cognition are also verbal. Formulating definitions, properly speaking,

[1] On the role of activity in child's mastering previous experience see A. I. Meshcheryakov. 'How Is the Human Psyche Formed in the Absence of Vision, Hearing, and Speech'. *Voprosy filosofii*, No. 9, 1968.

is in such cases usually reduced to the researcher looking for a suitable name to refer to the results of the description of the specific characteristics of objects; but sometimes one has to resort to direct manipulation of the objects to define the meanings of words, and these operations form part of the description of the mechanism of definition (ostensive definitions).

Ostensive definitions are definitions of the meanings of words and phrases through directly familiarising the subject with the objects, actions and situations denoted by these words and phrases. Ostensive definitions are thus definitions of words and phrases unfamiliar to the subject, in explaining which the teacher does not resort to the meanings of other words or phrases.

The teacher knows the meanings of the words and phrases to be explained to the subject, so that in this sense his language is richer than that of the subject. The explication does not proceed by reasoning but by demonstrating objects and situations, by operating with the objects, by acting with them and simultaneously naming the objects, actions and situations by appellations unfamiliar to the subject.

Using ostensive definitions is a necessary part of teaching a child its native tongue. An adult shows the child certain objects and names them, the objects recurring in various situations. Since signs and sign expressions prove to be more 'rigid', more stable than the objects denoted by them, ostensive definitions play an enormous role in developing the child's ability to identify with itself an object given an identical appellation or label, to identify different objects (since different objects are given identical or similar names), to distinguish various objects and their classes (since objects differing in some properties may be given different names). Ostensive definitions develop the child's ability to perform the identification abstraction. The meanings of words and phrases are learnt most productively in situations which involve the child's activity and satisfy its needs.

On the basis of ostensive definitions the child learns not merely the meanings of separate words but also of whole phrases ('drink your milk', 'this is a cat', etc.), learning in this way the grammatical patterns of his native language.

In definitions of this sort the Dfd is a sign expression, and the Dfn, the demonstrated material object that has not been registered in a linguistic form and is independent of the

Dfd. The schema for explicit semantic definitions 'Dfd'≡≡Dfn cannot therefore be applied to the description of the structure of the ostensive definition. Ostensive definitions are a mode of establishing correspondencies between signs and objects, with the result that the sign becomes meaningful for the subject. This seems to be the reason why many authors regard ostensive definitions as *protodefinitions*, not definitions proper.

Determining that to which the sign expression refers by means of ostensive definition is not always so trivial a task as might seem at first glance. For instance, we show the child milk in a bottle, glass, saucer, etc., and call it 'milk' (these appellations for the object are isolated words but rather corresponding elliptical sentences).[1] On the strength of the demonstration of the milkbottle and the name 'milk' as heard, the child cannot yet decide to what object the name refers—the bottle, the bottle with the milk in it, or that which may be poured out of it. Determining the invariant to which the word refers involvles the analysis of many different cases of the use of the word 'milk' and the child's analysis of its own activity accompanying the uses of the word. This is the only basis for singling out that invariant in the object demonstrated which is later associated with the word 'milk' as its meaning.

The question arises as to what may be assimilated through ostensive definition and in what cases one has to resort to verbal explanations through other words. Numerous words belonging to the various parts of speech are assimilated by man with the aid of ostensive definitions. Of course, scientific cognition of reality cannot succeed if it uses ostensive definitions alone—and that not only because science uses the enormous and complicated experience accumulated by past generations which has to be transmitted by means of language, but also because science, since its inception, has been using abstractions, idealisations and schemata which are not given in immediate sensual experience (cannot be directly 'demonstrated'). The meanings of such words as 'cat' or 'dog' can be established by means of an ostensive definition, but the meaning of such words as 'animal' can hardly be explained ostensively (cf. the situation in which

[1] See B. Russell. *Human Knowledge, Its Scope and Limits*. Allen and Unwin, London, 1948, Part II, Chapter II.

we have to apply one and the same name to man and some species of coral polyps).

Alonzo Church makes use of the term *implicit ostensive definitions.* He sees the need for them in the fact that there are words in language whose meanings are established ostensively but which are used in a certain context. Thus, one cannot point out directly 'here' and 'now', 'this' and 'that', 'before' and 'after', 'far' and 'near': their meanings are established through context (e.g., 'it is clean here and dirty there', 'milk is here and bread is there', etc.).

Many of the words that are given implicit ostensive definitions are sometimes called *demonstrative.* The distinctive feature of demonstrative words is that their meaning is in a measure changed with every new instance of their use, depending on who pronounces them, where and when. Of this nature is the word 'now'. Thus, the sentence 'It is warm now' may be true or false depending on the time and the place in which it is pronounced, because the meaning of the word 'now' changes with its correlation with certain temporal and spatial conditions. It is quite natural that these words are replaced in the exact sciences by temporal and spatial characteristics in definite systems of coordinates (e.g., a certain position of a moving body is described by one quaternion, another position—by a different quaternion, etc.). The observer registering the here-and-now characteristics of a moving body acquires the property of 'intersubjectivity', since the time and the place of observation are registered with precision, and the observer himself may in each situation be replaced by any other observer or even by an appropriate device, and this does not affect the readings which they communicate at any particular point.

Ostensive definitions do not determine the meanings of many words and phrases; moreover, the definitions of the words and phrases that *may* be arrived at in this way do not, as a rule, satisfy the requirement of nonambiguity or completeness. A child may, for example, assimilate ostensively the meanings of such words as 'fish', 'bird', which may be demonstrated by the fact that it applies these words to particular birds and fish in accordance with their general usage. Such coincidence, however, is usually limited to the experiential sphere in which the original assimilation of the meanings of such words took place, or under conditions which, superficially, closely resemble previous experiences. If, for

instance, the child assimilated the meaning of the word 'fish' through experiences with river fishes, on seeing a whale or a dolphin it will most probably call the latter 'fish' as well, meaning their external similarity with the animals it earlier called 'fish'. Such definitions are *incomplete* (see 1.14).

However, if we introduce a special limitation on the domain of ostensive definitions, restricting it to the sphere where it is most essential to teach a child to use words correctly ('the sphere of everyday life') and, correspondingly, to distinguish between objects and identify them, these definitions will be *complete in a certain conditional sense*, that is, with regard to those goals that are realised through these definitions. The incompleteness of ostensive definitions will in this case be essentially different from the incompleteness of some other kinds of incomplete definitions, e.g., from definitions the purpose of which is to indicate the genus that includes a certain species unfamiliar to the subject (cf. 'The whale is a great sea animal that looks like a fish'). These definitions are not ostensive but verbal (see 1.14).

A widening of the sphere of activity, assimilation of the information accumulated by mankind make the child see that a definition that seemed complete for some level of its knowledge and activity may prove to be in fact incomplete. The child's intellectual development, education and training involve changing definitions that prove to be incomplete. This process is many-staged: in a sense it reproduces the development of concepts and their definitions on the phylogenetic plane.

If a person has mastered a certain set of meanings of words and phrases by the ostensive mode, this 'ostensive' vocabulary may later serve as the basis for introducing a great number of the words into his vocabulary by means of verbal definitions. The Dfd in explicit verbal definitions will be a term unfamiliar to the subject, and the Dfn, a phrase the meaning of which has been assimilated ostensively or, in its turn, defined verbally.

At some point in the child's education and training the share of verbal definitions begins to grow constantly, and plays the principal role in its mastery of the accumulated experience of mankind. It is appropriate to draw the following parallel here. As is generally known, the language of a scientific theory includes a certain basic vocabulary where

the meanings of terms a e assumed to be familiar (they are not formally defined within the framework of the given theory but are clarified through examples, through a description of the logical and methodological procedures of their formation, etc.). The meanings of other terms of the theory are introduced by definitions. They become definite and familiar as they may ultimately be reduced to the meanings of the terms of the basic vocabulary. Introduction of terms into a scientific theory may be likened to the man's mastering of the accumulated knowledge.

In a similar way, a set of expressions of the natural language enriches the child's memory on the basis of experiential activity rather than through formal or purely verbal definitions. New meanings, concepts, or objects are later introduced by formal definitions on the basis of the meanings of the terms of the 'basic vocabulary'.

The parallel drawn here may serve as the basis for the following hypothesis. Just as one and the same scientific theory may be more productive and fruitful depending on the choice of the 'basic vocabulary',[1] assimilation of past experiences by a child, its subsequent development, direction and rate of progress of its intellectual and, more generally, psychological activity on the whole essentially depend on the kind of vocabulary (its extent and content) that will be formed in the child's memory on the basis of ostensive definitions, on the kind of the basic vocabulary of meanings assimilated ostensively, which will allow a successful transition to a more productive verbal assimilation of experience. This hypothesis, combined with an analysis of experiences in constructing scientific theories, may serve as the basis for staging useful pedagogical and psychological experiments.

Some analogues of ostensive definitions are also used in science. Physicists observe various effects in their experiments, describe the modes of obtaining them and name them. This method of introducing a name differs from the ostensive in that not only is a new name introduced verbally, but a verbal explanation is also given of the modes of obtaining the effects that are of interest for the researcher. Since a verbal explanation in itself does describe the Dfd unambigu-

[1] The choice of a 'basic vocabulary' for an empirical scientific theory assumes not only an analysis of the empirical stage in the development of science but also of the goals to be solved by the theory.

ously and even, taken in isolation, lays no such claims, this description cannot be viewed as a verbal definition. The goal of unambiguous description is attained here through combining verbal description and ostensive demonstration. These definitions are sometimes called *semi-ostensive*. Thus, in attempting to explain the Zeeman effect one may say: 'Take a look at that yellow line in the spectrometer, and now watch what will happen when I switch on the solenoid magnet; that is the Zeeman effect.' The so-called *exemplary* definitions (those through examples, see 6.4) are also close to ostensive definitions.

An interesting analysis of correlation between ostensive and verbal definition was given by K. Popa, who used some semiotic ideas.[1]

1.12. THE SO-CALLED LINGUISTIC AND CONCEPTUAL DEFINITIONS

Nominal definitions in natural language are sometimes subdivided into the so-called linguistic and conceptual definitions.

Linguistic definitions are taken to mean nominal definitions of unfamiliar words and phrases of a foreign language through dictionaries or information supplied by the teacher (e.g., *der Tisch*—'the table'; *la table*—'the table'). Other nominal definitions, intended to establish the meanings of terms within a given language, are called *conceptual*. We believe, however, that the so-called linguistic definitions should not be regarded as definitions at all. The point is that the meanings of words in the various natural languages are already defined—either contextually at the level of speech or explicitly at the level of language (e.g., in dictionaries). Translation of a word unfamiliar for a student might be interpreted as synthetic nominal definition, where the Dfd is an unfamiliar term of a foreign language, and the Dfn, a term of his native language the meaning of which he knows. However, the concept of definition in general presupposes that the analysis of the meaning of a term is undertaken within a concrete language (see 2.2).

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[1] See Kornel Popa. *Teoria definitiei*. Bucuresti, 1972, Chapter 3.

This limitation, in our view, is expedient because translation and determining its adequacy with regard to the original involve a number of procedures and rules that are irrelevant for definition.

The so-called *hortative definitions* are sometimes singled out among conceptual definitions. Hortative definitions are specifications of meanings of some incomplete expressions in a language introduced for the nonce to guarantee the correctness of certain reasoning. For instance, in discussing acceleration in the development and growth of man in these days, one may introduce the concept of 'tall person' by definition: 'A tall person will be taken to mean any person not less than 6ft tall.'

1.13. EVERYDAY AND THEORETICAL DEFINITIONS

All definitions may be subdivided into everyday and theoretical depending on whether our goal is to specify the meaning of a term in natural language, introduce a new term into natural language, or whether we attempt to perform the same operations at the scientific, theoretical level. In the latter case theoretical definitions form an essential part of scientific theories. The demands of rigorousness made on definitions in natural language and on scientific definitions differ considerably.

Definitions of word meanings in dictionaries may be regarded as examples of everyday definitions. Definitions in dictionaries (defining and phraseological) are introduced to explain expressions listed alphabetically. That is why we may somehow avoid the pitfall of circularity in defining the meaning of one word, that is, we eliminate the Dfd (if it is a simple term) from the Dfn; but we cannot solve the problem of circularity for the whole of the defining or phraseological dictionary. We list alphabetically, step by step, ever new terms for the Dfd, define them through some of the expressions for the Dfn, and then in defining the terms occurring in the Dfn we again use the Dfd terms already defined through them. In doing this, we endeavour to formulate unambiguous definitions for words and phrases with different meanings—naturally, at the level of experience accumulated in the natural language. Formulating unambiguous definitions is

considerably facilitated by citing the various contexts in which the Dfd term is used.

Strict formal demands on the technique of definition become less important than the demand of attaining intuitive clarity of the sense or meaning of the Dfd term in formulating everyday definitions.

Theoretical definitions constructed in scientific knowledge may be subdivided into several classes depending on the level of knowledge. We shall point out just two such classes.

The first class characterises the level of empirical knowledge (meaning experimental sciences). At this level, scientific terminology is created within the framework of a certain scientific discipline. Homonyms are excluded from scientific terms. As the domain of study is specified and objects described on the basis of data obtained in experiment and observation, conditions arise for interchangeability of the Dfd and Dfn terms in a more rigorous sense than in the case of everyday definitions.

Avoiding circularity, however, is a very difficult problem within the context of a certain scientific discipline as a whole, since at this level of cognition the 'basic vocabulary' of the science, i.e., the stock of its undefined primitive terms, is not yet established, as a rule. At the theoretical level (the second class of theoretical definitions) definitions are regarded as an integral component of the theoretical system.

Defining of concepts and objects under study becomes much more ordered at the level of mathematical theories and the theories of mathematical natural sciences. Here a class of primitive terms is selected whose meanings are usually explained (to avoid circularity) independently of the meanings of terms emerging in the step-by-step development of the theory. The meanings of primitive terms are explained through examples, the concepts of other sciences, through describing the mode of constructing these concepts, etc. These explanations are usually viewed as lying beyond the closed theory. Introduction of new objects and concepts as well as corresponding terms on the basis of definitions within a closed theory is ultimately rooted in a certain stock of the meanings of primitive terms. We encounter such ordered systems of definitions not only in the theories of mathematical natural science but also in a number of biological and

social theories, which are on the whole characterised as qualitative descriptions of the domain of reality under consideration.

1.14. COMPLETE AND INCOMPLETE DEFINITIONS

The subdivision of explicit definitions into complete and incomplete is essential for scientific methodology. *Complete* definitions are ones satisfying the equal extent requirement, resp. the requirement of interchangeability of the Dfd and the Dfn in standard contexts. *Incomplete* definitions do not satisfy the above requirement.

We sometimes resort to incomplete definition when (a) the addressee is not ready to understand the corresponding complete definitions; (b) the incompleteness of definition is the property of definitions themselves and is determined by the contexts in which the latter occur.

Incomplete definition in a situation where the addressee is unable to comprehend the corresponding complete definition may be illustrated by definitions addressed to a child or person without proper training in some field or other. For instance, in reply to the child's question, 'What is ostrich?', the father may say, 'An ostrich is a large bird that cannot fly'. Bearing in mind the addressee's experience, it may be assumed that another, more complete and scientific definition would not be suitable here. Functionally, this sentence is a definition but, generally speaking, an incorrect definition. Ostensive definitions considered above belong here (see 1.11).

Incomplete definition of the second type may be illustrated by recursive definition for which there is no algorithm permitting to determine the values of the function for all the values of its arguments (the general recursive function is not a completely defined function), as well as by definitions by means of systems of axioms.

Incomplete definition of the same type may be illustrated by the so-called exemplary definition *(exemplarische Bestimmungen)* of indefinable terms (resp. concepts) in meaningful theories constructed, e.g., axiomatically. These definitions precede the construction of a theory and are based on our intuition, experience, the knowledge of other sciences. The development of scientific knowledge is hardly feasible with-

out such definitions. One can hardly imagine that the development of geometry could have begun with those concepts of point, straight line, and plane, whose properties and correlations of properties are only implied in the relevant axioms.

Analysis of incomplete definitions is of considerable interest within the philosophical-methodological and pragmatic approaches. In traditional logic, they are often treated as procedures used instead of definition.

* * *

We have not exhausted all kinds of definitions occurring in natural language and various scientific theories, having concentrated primarily on those types of definition that occur most frequently in most diverse sciences and in the discussion and solution of most diverse problems. The description of definition types contained in the present chapter is the basis for considering the problem of what the operation of definition is in general, as well as the problem of the rules of definition.

CHAPTER 2

ON DEFINING DEFINITION. RULES OF DEFINITION

2.1. THE VARIOUS APPROACHES TO DEFINING THE TERM 'DEFINITION'

The meaning of the term 'definition' (ὁρισμός, *definitio*) [1] is not unambiguous. Different authors give differing answers to the question of what procedures may be called definitions. In discussing definition, one usually has in mind both the logical procedure and the result of this procedure registered in an appropriate sign expression.

Regardless of the solutions as to what is defined in definition, there is no unanimity concerning the kind of procedure and the corresponding propositions to be included among definitions. This lack of agreement prevails both among logicians and among specialists in the various sciences.

When definitions are interpreted as real, the question frequently arises whether any definition of an object through its distinctive or specific features should be regarded as its definition or whether definitions should include only those unambiguous descriptions of the defined object which indicate its essential distinctive features.

Within the conceptualist approach to definition this same question was posed in the following form: is it justifiable to restrict oneself, in defining a concept well fixed, to listing its inessential properties, though they might unambiguously single out an object correlated with the concept? Won't we then merely establish the meaning of the word corresponding to the defined concept instead of defining the concept? Can one, in general, consider all the extensionally equivalent

[1] ὁρισμός comes from the Greek ὅρος which means 'boundary post', the sort of posts that were used to mark one plot from another. *Definitio* comes from the Latin *finis*, which also means 'boundary', 'end of something'. The Russian *opredeleniye* comes from *delit'*, 'to set a limit, draw a boundary line'.

but unambiguous descriptions of one and the same Dfd to be its definitions? Would it not be expedient to include only scientific definitions in the class of definitions? Different answers have been given to these questions in the history of logic.

Some logicians formulated this question in the following manner: must the description of the definiendum in a definition be necessarily unambiguous? Can so-called axiomatic definitions, definitions through abstraction, incomplete explicit definitions of objects be regarded as definitions at all?

In discussing these and similar questions, H. S. Leonard takes a very broad view of definition. Thus, he includes among definitions explanations of sign descriptions of this sort: '*Tête-de-pont* is a particular kind of bridge-head', if this expression is, for instance, a teacher's answer to a student's question as to what a *tête-de-pont* is (this answer indicates the kind of objects to which a *tête-de-pont* belongs, but does not provide an unambiguous definition). Leonard calls such descriptions incomplete definitions [1] (see 1.14).

It is a matter of constant concern to logicians whether the class of definitions should be restricted to procedures that do not go beyond the limits of intellectual and sign activity, or whether definition should also include procedures which, apart from resorting to intellectual and sign activity, involve going beyond this sort of activity into the realm of object activity in each particular case. Authors adhering to the first point of view decline to grant the status of definition, for instance, to the so-called ostensive definition, calling it protodefinition (Joseph Bochenski). Others do include ostensive definitions in the class of definitions (e.g., Leonard subdivides all definitions into ostensive (indicative) and verbal, see 1.11).[2]

Let us now assume that we have restricted the class of definitions to explicit verbal definitions appearing, at that, only in the sciences. Even this strong assumption would not ensure unanimity among representatives of the various sciences as to whether all or only some of the narrow range of propositions indicated above are definitions in the proper sense of the word.

[1] See H. S. Leonard. *An Introduction to Principles of Right Reason*, pp. 273-75.
[2] *Ibid.*, pp. 313-14.

Thus, definitions that in most experimental sciences are regarded as definitions in their own right without any reservations, satisfying all the requirements of scientific rigorousness, are frequently treated in logical and mathematical disciplines as explanatory propositions, as descriptions. Of this nature are, for example, the definitions of algorithm formulated on the basis of natural language, those of correspondence, set, function, variable, letter of alphabet and other primitive mathematical concepts. They are described with the aid of words whose meanings are supposed to be clear from everyday experience and are explained through example.[1] At the same time in constructing rigorous mathematical calculi some primitive elementary objects are introduced without definition at all. As for the other complex objects of a theory (e.g., a certain number or the values of dependent variable of some function), they are given definitions, the relevant procedure frequently being reduced to indicating an algorithm for constructing such a number or for finding the values of such a function. The mathematical object is in this case defined by applying a set of rules to other objects accepted without definitions.

Definition in mathematics is regarded as satisfying the mathematical criteria of rigorousness if, apart from the general logical requirements, it also satisfies a number of additional conditions determining the direction and boundaries of mathematical reasoning, which is aimed at obtaining new results and ensuring their strictness and effectiveness.

Let us formulate in most general form those additional conditions which, if we add at least one of them to the general logical requirements, will elevate the definition to the rank of a properly mathematical one, satisfying mathematical conditions of strictness.

(a) A definition incorporated in a mathematical theory or calculus should permit application of strict formal rules to the objects introduced by the definition; it is used not only to extend the language of the theory but also in solving problems, in proving the theorems of a theory, and in formulating new rules. What is more, all the characteristics indicated in definitions must be used in the mathemati-

[1] See V. A. Uspensky. *Lektsii o vychislimykh funktsiyakh* (Lectures on Computable Functions). MGU Publishers, Moscow, 1960, p. 16.

cian's work. Some of the definitions listed in the preamble to the first book of Euclid's *Elements* illustrate the case when definitions of a meaningful mathematical theory do not satisfy this requirement: 'A point is that which has no parts', 'A line is length without width', 'A straight line is a line that is equally situated with regard to the points lying on it', etc., as well as the propositions defining surface and plane. The fact is that in the whole of the 13 books of the *Elements* Euclid does not resort to these definitions in developing geometry, in proving the theorems concerning the properties of other geometrical objects. V. F. Kagan points out quite correctly that Euclid's concepts of point, straight line, surface, plane should be regarded as basic concepts introduced without definition.[1] That means that the above definitions cannot be viewed as mathematical ones. At some other level, however (e.g., that of pregeometric reasoning), they may be regarded as definitions in their own right. (Let us note that at this level, too, the definition of straight line cited above is also defective in that it indicates a property not only of straight lines but also of circles and spirals).

(b) Explicit definitions of objects of which it is assumed that they are somehow constructed, pre-exist the theory and ought to permit the deduction of all the other properties of the objects (i.e., the proving of all the theorems concerning their properties) on the basis of rules, the basic propositions of the theory and theorems proved earlier. Of this kind are, for instance, definitions of circle, square, and triangle in Euclidean geometry. However, definitions that do not satisfy the additional requirement (b) may at some stages be admitted in mathematical theories. Thus, the definition of a continuous curve as one that may be drawn without taking pencil from paper (Descartes' definition) can hardly yield such continuous curves that do not have tangents at any of their points (the Weierstrass-Bolzano theorem). This latter task requires a more exact definition.

In the formalised theories of logic and in the formal systems in general one strives to lend still stronger constructive character to the formulations of definition rules; new demands are formulated on definitions and a number of

[1] See V. F. Kagan. *Osnovaniya geometrii* (Fundamentals of Geometry). Akademiya Nauk SSSR Publishers, Moscow-Leningrad, 1949, p. 42.

limitations imposed on the rules of operating with objects, resulting from definition, in the various formalised systems of logic.[1]

All this means that elaborating a general definition of what should be meant by definition runs into considerable difficulties. The choice of some definition satisfying a certain criterion of rigorousness is primarily conditioned by the goals of cognitive activity and other circumstances (e.g., the level of knowledge or the nature of constructing a scientific theory).

2.2. ON DEFINING DEFINITION IN THE BROAD AND THE NARROW SENSES

We have seen that the logical procedures called definitions in scientific and linguistic practice differ essentially from one another. The problem of defining the concept of definition, however, can hardly be reduced in that case to listing and describing the procedures that are called definitions in ordinary language and scientific practice.

The situation here seems to be as follows.

The linguistic use of the term 'definition' follows a fixed pattern. A natural desire arises to elaborate a definition of definition that would embrace all the cases of linguistic use of the word 'definition', that is, to elaborate a corresponding analytical nominal definiton of this term. That means that the class of objects to be defined pre-exists definition. The problem is to discover the common and essential characteristics in the elements of this class and include them in the corresponding real definition. It may (and did) happen in the history of science that the same term is used to denote so diverse objects that some of the subclasses have to be subsequently excluded from the class corresponding to the Dfd, which entails a splitting of the concept introduced by the definition.

However, all definitions without exception have one common goal and solve (or attempt to solve) one common cognitive task. Assuming that all definitions have been translated

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[1] See J. Slupecki, L. Borkowsky. *Elementy matematicheskoi logiki i teoriya mnozhestva* (Elements of Mathematical Logic and Set Theory), Progress Publishers, Moscow, 1965, pp. 140-55 (Russian edition).

into nominal ones (and we have seen that this translation is possible, see 1.2), the definition of definition in the most general methodological sense (meaning the common goal or task solved by any definition) may be formulated like this. *Definition is an intellectual method aimed to establish, specify, or explain the meaning of a sign expression in a certain language* S *or to extend the language* S *by introducing new sign expression.*

Thus, if such a goal exists we shall be dealing with definition *in the broad sense.* In light of the definition given here, definition will include ostensive definition, too, as the latter is used to extend and enrich the learner's language. Among propositions encompassed by the definition above and regarded as definition one may come across the so-called *incomplete definitions*, i.e., those that do not satisfy the requirement of interchangeability of the Dfd and the Dfn for some reason or other.

We have already pointed out that incompleteness of definition may be due to a number of reasons (see 1.14). Thus, the primitive terms of a meaningful theory cannot be explicitly defined within the framework of the theory itself, for fear of incurring *regresus in infinitum.* In such case we restrict ourselves to citing examples ('exemplary definitions'), describing the meanings of these terms on the basis of natural language and on the basis of terms and concepts borrowed from other sciences. Apart from this, all definitions elaborated in science by generalisation and analysis of empirical experience may be viewed as complete only under fixed historical conditions, as our knowledge of objects changes and develops continually, and it may so happen that a definition regarded as complete (we mean extensional equality of the Dfd and the Dfn) will in the course of time prove to be incomplete—either too broad (when Dfd$<$Dfn) or too narrow (when Dfd$>$Dfn). The same is true, *mutatis mutandis*, of definitions in mathematics (cf. the historical changes in the definitions of variable, continuous curve, number, etc.).

Incompleteness of definition, as has been pointed out, may be a function of a particular structure of definition determined in its turn by the structure of the corresponding theory (cf. implicit axiomatic definitions). Incompleteness may also be determined by certain didactic considerations, e g., the addressee's level of development.

The definition of definition in the proper (logico-semantic) *sense* (bearing in mind that real definitions have been transformed into nonsemantic nominal ones) may be formulated as follows: *it is an intellectual device designed not merely for determining, specifying, and explicating the meaning of a sign expression or extending a certain language* S *by introducing a new sign expression, but also for satisfying the requirement of interchangeability of the Dfd and Dfn with regard to the corresponding propositions of the language* S, *provided the structure of the definitions themselves and the level of social knowledge permit the satisfaction of these requirements.*

We may thus say that definition in the proper logico-semantic sense is the correct part of definition in the broad methodological sense. It will not include, for example, incomplete and ostensive definitions. The definition given here requires certain qualification and explanation.

(1) It is assumed that we are capable of reasonable abstraction from the historical incompleteness of our knowledge that goes into the formulation of the corresponding definition, and of formulating a definition satisfying the demand of completeness at the given stage in the development of knowledge. Definitions constructed on the basis of such abstractions serve as the foundation for scientific theories. The development of our knowledge often makes us rebuild and perfect the theories created, which involves perfecting previously introduced definitions.

(2) The definition of definition in the proper sense given above covers all cases of explicit complete definitions, axiomatic implicit definitions (since their incompleteness is determined by the structure of definitions themselves which form part of the axiomatic theory), as well as all cases of implicit definitions which may be reduced to complete explicit definitions, and those explicit definitions whose incompleteness is determined by the conditions indicated above.

(3) The very concept of explaining or specifying a sign expression already existing in the language S presupposes a reduction of the less known (the *explicandum*) to the better known. This is only possible when a new term is introduced through other terms whose meanings are already known and defined, and when the meaning is specified and renovated through something already familiar and understandable and ultimately through that which is accepted as something

primitive and basic within the framework of a given theory.

When we deal with definition in natural language, it is impossible to determine strictly enough and without limitations on generality which terms and expressions are more understandable and which are less understandable. What may seem more comprehensible to some person will appear less so to others. The concept of definition itself acquires a more precise meaning and the basic cognitive task of definitions becomes more clearly formulated when the process of definition is analysed with reference to certain scientific theories, scientific conceptions, scientific reasoning of a more or less complete nature. It seems possible to single out in these cases that from which we proceed as from the familiar and, consequently, to single out the stock of terms the meanings of which are deemed already established, familiar. This stock serves as the basis for introducing new terms by means of synthetic definitions and for specifying terms earlier introduced into the language S by analytic definitions.

The same conditions help to avoid vicious circle. As for definitions in defining dictionaries, they are circular due to the reasons indicated above (see 1.13), but they cannot be regarded as containing a vicious circle since the very type of ordering the words of some language in a dictionary *inevitably* involves circularity; we do not set ourselves the task of establishing the kind of sequence of introducing terms into the language which we find in closed and sufficiently rigorously constructed scientific theory.

2.3. RULES OF DEFINITION

Meaningful scientific definitions must satisfy a number of requirements that are known as rules of definition. These requirements may be of three kinds, *literary*, *factual*, *and logical*.[1]

Literary requirements usually mean that *definition must be as clear as possible, and must avoid figurative and metaphorical expressions.* This implies that the meaning of terms used in the definition should be sufficiently clear for the addressee. Only in this case can the meaning of the term explained or introduced by definition be revealed. The scien-

[1] See H. S. Leonard. *Op. cit.*

tific and, in general, intellectual level of discourse should be free from figurative and metaphorical expressions, which are specific for other types of discourse (e.g., for fiction). The fact is that in artistic or poetic language the meaning of a certain expression as a whole is essentially different from its meaning in scientific or even everyday language. What is quite acceptable in the former becomes absurd when translated into the latter. Moreover, in fiction the object is not described on the basis of its really essential features but on the basis of an image created by artistic linguistic means. Thus, if we tried to give a scientific interpretation of the definition 'Architecture is music turned to stone', we would naturally have to interpret any architectural structure as emerging from music through certain procedures or transformations. We would therefore have to recommend selecting certain musical works, performing them and subsequently embodying them in architectural structures, which is patently absurd.

The need for clarity and unambiguity of terms was emphasised already by Aristotle. He wrote: 'We may add that if dialectical disputation must not employ metaphors, clearly metaphors and metaphorical expressions are precluded in definition: otherwise dialectic would involve metaphors.'[1]

The factual demands made on definitions include the following:

1. In real definitions (including those that may be obtained by transforming nominal definitions), *the Dfd must be singled out and specified by essential features.* Correspondingly, in nominal definitions involving specification of a term already existing in the language, or introduction of a new term, the intensional characteristics of meanings explicitly registered in the Dfn must be essential. Application of this rule is fraught with certain difficulties, as there is no criterion sufficiently rigid to permit the general distinction between the more essential characteristics and the less essential ones. Still, there is a possibility for solving this problem through taking into account the goals of meaningful theories, conceptions and arguments, the nature of problems being solved, the level of knowledge attained, and by relying on

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[1] *The Works of Aristotle.* Vol. I. 'Analytica Posteriora', Book II, 13.

past experiences. This requirement therefore plays the role of a certain principle guiding the searching and cognitive activity. We must bear in mind, however, that the distinction between the essential and the inessential becomes meaningless in mathematics: if we succeeded in singling out a certain object of theory by definition, the addition of this definition to the axioms, theorems proved earlier, and definitions already introduced would enable us to prove just the same theorems—irrespective of the nature of properties specifying a certain object. Such definitions are deductively equivalent in this sense.

The reason seems to be that mathematics, as distinct from experimental sciences, singles out only the essential ignoring everything that is inessential in forming and operating on abstract object (see 8.1 for detail).

2. *A term already introduced into some language must be specified or clarified through terms whose meanings are already known, more understandable and clear than the meaning of the term being specified.*

When a new term is introduced within the framework of a rigorous theory (for instance, an axiomatic one, where explicit definitions introduced one after another are synthetic), this rule becomes superfluous, as the new terms are introduced here through a certain set of primitive terms whose meanings are in a sense revealed in the axioms, or through previously defined terms. In nonrigorous reasoning at the natural language level one has to comply with this requirement, although it is not constructive. Definitions in textbooks and manuals presenting the fundamentals of certain sciences in compressed, abbreviated form must also satisfy this requirement by all means.

The requirement must also be taken into account in defining the basic concepts of sciences. These include, e.g., 'number', 'function', 'straight line', 'figure', 'set'—in mathematics; 'length', 'mass', 'time'—in physics (these are the primitive concepts accepted without definition within a rigorously constructed closed theory and nonrigorously defined outside the theory). Thus, the Frege-Russell definition of natural (quantitative) number (see 1.8) has the drawback that it defines intuitively clearer objects through set-theoretical concepts that can hardly be regarded as clearer or more understandable than the concept of natural number itself.

We shall formulate the *logical* requirement with regard to explicit definitions.[1]

1. *The rule of interchangeability* (or elimination): in definitions of the *Dfd*=*Dfn* structure, the definiendum and the definiens *may be substituted for each other in any standard contexts.* This rule is sometimes called the rule of *translatability.*

This basic rule requires some explanations and specifications.

a) With regard to real definitions, where the Dfd and Dfn terms both appear in the use function *(Dfd*=*Dfn)*, this rule is usually formulated as the rule of proportionality (equal extent or equal interpretation of the Dfd and the Dfn). That is why in a language where the principle of extensionality holds, the Dfd and Dfn terms become interchangeable (see 1.1).

b) Application of the rule of interchangeability to nominal semantic definitions in which the Dfd term is employed in the function of mentioning and the Dfn term, in the use function (*'Dfd'*=*Dfn*), is conditioned on a preliminary transformation of these definitions in one of the two directions (see 1.2): both the Dfd and the Dfn terms must be either used (we shall then obtain a real definition: *Dfd*=*Dfn*) or mentioned (we shall then obtain a nominal definition of the form *'Dfd'*=*'Dfn'*). In the first case the Dfd and Dfn terms will become interchangeable owing to equal extent; in the second, owing to identical meaning. When the rule of interchangeability is formulated as proportionality rule, as in most books on traditional logic, its domain of application is in general narrower than that of the interchangeability rule formulated above: there is no sense in applying it to any but analytic real definitions. The rule of proportionality is not directly applicable to synthetic definitions, which are semantic definitions. Neither is there any sense in applying this rule to such definitions after they have been transformed into real ones, as conformity with this rule is ensured by the structure of synthetic definition itself.

c) When the term Dfd is introduced in *meaningful* theo-

[1] The traditional logical requirements concerning definitions *per genus proximum et differentiam specificam* have been thoroughly analysed in A. Dumitriu. 'Les conditions de la definition'. *International Logical Review, Rassenda internationale di logica*, No. 1, Bologna, 1970.

ries, arguments or conceptions as an abbreviation for a more awkward description corresponding to the Dfn, the Dfd and Dfn terms become interchangeable on the strength of (b): these terms are either equal in extent, as they are used to denote the same thing, or else they have the same meaning.

d) The rule of interchangeability (or elimination of the Dfd term by replacing it by the Dfn, or of their mutual translatability) is usually formulated with regard to all contexts. Here we mean most varied contexts occurring within certain fixed theories, arguments, conceptions, and it is assumed that we can differentiate between the various meanings of identical sign expressions in different sciences. It is a fact, however, that identical sign expressions are sometimes ascribed different meanings in different sciences (cf. 'mass' in physics and in the social sciences, 'reaction' in chemistry, physiology, and the social sciences). However, in applying this rule to texts belonging to the same domain of knowledge, certain limitations have to be introduced too.

Let us consider an example.

Assume that we have a definition of 'less':

$$((x < y) \rightleftharpoons \exists z\,(z \neq 0 \wedge y = x + z)).$$

This definition being valid for any x and y, it is also valid for some α and β. Then

$$\alpha < \beta \rightleftharpoons \exists z\,(z \neq 0 \wedge \beta = \alpha + z). \qquad (1)$$

Assume that we are given the context: 'The expression "$\alpha < \beta$" has only three signs: α, $<$, β.' We cannot substitute the Dfn for the Dfd in this context as this will result in a false context instead of a true one: 'The expression

"$\exists z\,(z \neq 0 \wedge \beta = \alpha + z)$"

comprises only three signs: α, $<$, β' is false.

It follows that one cannot replace sign expressions for the Dfd by sign expressions for the Dfn and vice versa in contexts formulated at the level of metalanguage.

In a similar way, the so-called nonextensional contexts must also be excluded. Let us illustrate this by an example. Let us assume, for example, that we have a definition 'A square is a rectangle having equal sides' and the context 'He does not understand that a square may be regarded as a rectangle having equal sides'. In this context, one cannot substitute the Dfd for the Dfn or vice versa. If the first context

is true, the contexts resulting from substitution may prove to be false. (cf. 'He does not understand that a square may be regarded as a square', 'He does not understand that a rectangle having equal sides may be regarded as a rectangle having equal sides'). Wishing to retain the general nature of the basic rule, we therefore introduce the concept of *standard context* in the formulation, which is taken to mean diverse meaningful contexts of various fixed theories, discourses, conceptions, except nonextensional and metalinguistic ones of the type indicated above. If we come across the expression $\alpha < \beta$ in a discourse formulated in an object language, we can replace it by the expression which it equals by definition, namely,

$$\exists z\,(z \neq 0 \wedge \beta = \alpha + z).$$

The truth-value of the context will in this case be retained, as we are dealing with a standard context here.

2. *The rule for excluding the vicious circle.* This problem will be considered, on the one hand, in regard to scientific theories, discourses and conceptions (i.e., scientific contexts) and, on the other, in regard to separate definitions occurring not only in science but in everyday language as well. In regard to scientific contexts this rule should be formulated as follows: *If a new term (i.e., the Dfd term) is introduced instead of some complex description (i.e., the Dfn term), no term in the complex description can be introduced earlier or explained later through the term introduced.*

We shall thus incur a vicious circle in our reasoning if we define logic as the science of rules for constructing any reasonable discourse, and it turns out later that reasonable discourse has already been defined or will later be defined again as discourse constructed according to the rules studied by logic. Then again, if truth is defined as correct reflection of reality, and later correct reflection of reality is not defined independently of the concept of truth but through that very concept (e.g., correct reflection is that reflection of reality which is true), the discourse will contain a vicious circle. In other words, the Dfd term and other expressions defined through it should not appear in the Dfn. If they do, one falls into an error similar to the vicious circle in proofs, where theorems that serve as the basis for proving a given theorem are included in this theorem (e.g., in a different formulation) or are in their turn proved on its basis.

This rule is equally applied to *absolute* explicit definitions (e.g., in the examples cited above) and *nonabsolute* ones (see 1.4). Thus, if the term 'decaying oscillations' is introduced into a theory, nonabsolute real definition will have the form: 'Decaying oscillations are those which... .' The terms used in the Dfn must not be defined through the Dfd term, that is, through the term 'decaying oscillations', before the above definition is introduced.

In finding out whether definitions are circular or not, one should bear in mind that in constructing a rigorous scientific theory new terms (resp. objects) are introduced on the basis of primitive ones, not defined explicitly within the framework of the theory. The primitive terms and the newly introduced ones serve as the basis for introducing still newer terms, etc. In other words, new terms must be introduced into the theory only on the basis of those introduced earlier. Thus in classical mechanics such physical quantities as *mass*, *length*, and *time* are frequently taken to be the primitive ones, and the following units are chosen as the basic ones: *gram*, *cm*, *sec*. These units may be used to define *speed* (*cm/sec*), *acceleration* (*cm/sec*^{2}), *force* (*g·cm/sec*^{2}—*dyne*), *work* (*g·cm*^{2}*/sec*^{2}—*erg*), *power* (*g·cm*^{2}*/sec*^{2}—*erg/sec*), etc. These units are defined contextually, that is, through corresponding formulas (see Chapter 4). We observe that all objects newly introduced by definition may be expressed through the primitive ones.

Further, if we define power as work performed in 1 *sec*, it is not merely the term 'work' that must not be defined through the concept of power in the given system: no concepts earlier introduced into the theory can be defined through power either.

In regard to individual definitions the rule for excluding circularity is formulated as a particular case of the rule formulated above, namely, *definition must not be tautological*, that is, the term Dfd must not occur in the Dfn. This rule is violated in the following example: 'A democrat is a person of democratic convictions.'

When definitions are formulated on the basis of natural language, one may get the impression that they contain a vicious circle, although in actual fact that is not the case. Assume that we have the definition: 'To solve a problem means (=) to reduce its solution to the solution of other problems that we can solve or accept as solved'. One may

get the impression that the Dfd is here defined through the Dfd: the expression 'solution of the problem' recurs in the Dfd and the Dfn. In real fact, however, the meaning of the term 'solution of (some) problem' is here clarified through the meaning of the expression 'to reduce its solution to the solution of other problems.'

Definitions containing a circle that is not vicious are regarded as definitions in their own right. We spoke of them in 1.9.

3. *The rule of partial identity of the structures of the Dfd and Dfn terms.* This rule calls for the following:

a) If the Dfd has the form of a proposition formulated in some language, the Dfn must also have the form of a proposition. For example, 'To solve a problem means ($\equiv$) to reduce its solution to the solution of other problems that we can solve or accept as solved', 'A formula of propositional logic is tautologically true if and only if it is true for all the sets of its variables', 'Information is redundant if and only if...';

b) if the Dfd is a term (absolute or nonabsolute), the Dfn must also be a term and not a function or proposition. Most explicit definitions are of this nature;

c) if the Dfd is a certain propositional function, the Dfn must also be a propositional function with the same free variables. For instance, 'x is the grandfather of $z \equiv x$ is the father of one of the parents of z';

d) if the Dfd is an indicator function, the Dfn must also be an indicator function. For example:
$n! \equiv 1 \cdot 2 \cdot 3 \cdot , \ldots , \cdot n$.

4. *The rule of nonambiguity.* This rule is formulated with regard to definitions taking into account the scientific contexts in which the definition occurs. It should not be formulated with reference to individual definitions expressed in scientific language, since requirements of this rule are already contained in the interchangeability rule. With regard to definitions considered in the context of a scientific theory the rule should be formulated like this: *there must be only one Dfd functioning as a scientific term within a theory corresponding to each Dfn, but not vice versa, so that there may be a number of Dfn for each Dfd functioning as a scientific term.*

Thus the scientific term 'square' in Euclidean geometry may be the definiendum for a number of definienses in defi-

nitions. We may give different definitions of square depending on our goals: 'A square is a parallelogram in which all sides are equal and all angles are right angles', 'A square is a rectangle having equal sides', 'A square is a rhombus having equal angles', etc. Each of the definienses contained in these definitions has only one corresponding scientific term 'square'.

This rule regulates the construction of scientific terminology, of the language of a scientific theory. Application of this rule in constructing a scientific theory eliminates homonymy from the language of the theory.

The fact that there may be a number of Dfn's corresponding to one Dfd means, first, that expressions of the *Dfd*$\equiv$ $\equiv$*Dfn* type occur and, second, that for a single Dfd one may construct a number of expressions of the form *Dfd*$\equiv$*Dfn*$_1$, *Dfd*$\equiv$*Dfn*$_2$, *Dfd*$\equiv$*Dfn*$_3$, etc. These are the forms in which synonymity is manifested in theories. There is thus a one-to-many correspondence and not a one-to-one correspondence between the Dfd and the Dfn. However, the Dfn$_1$, Dfn$_2$, Dfn$_3$, etc., may be identified with one another from the point of view of their extensional. One and the same sign expression cannot therefore be introduced as a terminological element within a science to denote different objects (what is meant here are primitive terms and those introduced instead of complex descriptions, which may ultimately be reduced to combinations of identical primitive terms). But the Dfn's differing from each other intensionally, may correspond to one and the same term in the context of definition. Using Frege's terminology, one scientific term must always have the same meaning but it may have different senses.

These differences in the Dfn's for one and the same Dfd may be eliminated within a rigorous theory, which proves not only their extensional equality, but to a degree their intensional equality, too. Dfn$_1$, Dfn$_2$, Dfn$_3$, etc., differing intensionally may be represented by an identical combination of primitive symbols. A necessary condition for proving the intensional equality of the Dfd and Dfn is listing all the primitive undefined terms from the very beginning. Even the most rigorous theories are sometimes constructed in such a way that the stock of undefined terms is continually extended along with the development of theory, the extension not always being explicitly noted.

We shall cite just one example of establishing intensional equality of the Dfn and the Dfd. Assume that we choose the following magnitudes as the primitive undefined concepts for describing the system of classical mechanics' units: length (*cm*), mass *(gram)*, time *(sec)*. The magnitudes are here viewed as linked with the units of measurement.

Work may be defined as a magnitude proportional to the force applied to a body and the path travelled by this body $(A=F\cdot s)$. However, work can also be regarded as a magnitude proportional to body mass, its acceleration, and the path travelled by this body $(A=m\cdot a\cdot s)$. Expressing these definitions in the primitive concepts (magnitudes), we shall observe that our definitions are intensionally equal: the Dfn will in this case be represented by an identical expression: $g\cdot cm^2/sec^2$.

5. *A definition must not be contradictory.* With regard to explicit definitions of the *Dfd≡Dfn* type this rule means:

First, there must be no contradiction in each separate definition isolated from the context in which it occurs.

Second, if addition of a noncontradictory definition to a closed deductively constructed theory results in a contradiction, two alternatives are possible. The definition introduced is defective and should be rejected and replaced by another. The new definition, determined by experience or reasonable considerations, will require a change in the theory itself (including changes in the definitions already introduced in it). Such cases frequently occur in rigorously constructed theories of the mathematical natural sciences (e.g., in physics). Situations like this may arise not only in rigorously constructed theories but they may also arise at various levels of empirical cognition, in various documents recording experiences (cf. the various legal codes constantly supplemented by new statutes and resolutions).

Implicit axiomatic definitions considered jointly with the rules of the system must also be consistent. The methods for establishing consistency of axiomatic systems have been elaborated in mathematics and mathematical logic.

Contradictory definitions in set theory may be illustrated by the definition of the normal set, i.e., the set of those sets that do not contain themselves as their elements. Thus, the set of cows is not a cow and cannot therefore be included in the set of cows (that is, the set of cows is a normal set). The set of concepts, however, is in its turn a con-

cept and must therefore be included in the set of concepts (that is, the set of concepts is not normal).

Let the variable f have sets and corresponding predicates for its values. Let F be the predicate 'to be a normal set'.

Then the definition of the normal set may be written in symbolic form as follows: $F(f) \sim \neg f(f)$, where $\sim$ is an equivalence sign.

This definition holds for normal sets only: its right and left part are both true when f is a normal set or the corresponding predicate. Thus, if we substitute 'cows' for f, then $\neg f(f)$ is true (it is false that the set of cows is a cow is true). The left part is therefore also true, and this set is included in the class of normal sets. Its right and left parts become false if, for instance, the set of concepts is substituted for f.

Since the expression $F(f) \sim \neg f(f)$ is true for any normal set and the corresponding predicate, it is also true for F.

By substituting F for f we shall then obtain:

$$F(F) \sim \neg F(F).$$

This formula is apparently a contradiction, which is easily proved from the following reasoning: $F(F) \sim \neg F(F)$ is equivalent to $(F(F) \supset \neg F(F)) \wedge (\neg F(F) \supset F(F))$ is equivalent to $(\neg F(F) \vee \neg F(F)) \wedge (\neg \neg F(F) \vee F(F))$ is equivalent to $\neg F(F) \wedge (F(F) \vee F(F))$ is equivalent to $\neg F(F) \wedge F(F)$.

To eliminate the possibility of such paradoxical contradictions, different authors suggest some limitations on the corresponding logical apparatus (for instance, Russell developed for this purpose the theory of types—a kind of calculus in which such paradoxical situations are excluded).

Another example of contradictory definition (a recursive one) is considered in 3.4.

CHAPTER 3

DEFINITIONS IN FORMALISED THEORIES AND FORMAL SYSTEMS

3.1. ON THE SPECIFICITY OF DEFINITIONS IN FORMALISED THEORIES AND FORMAL SYSTEMS

In axiomatic construction of formalised theories and formal systems primitive sign expressions of the theory or system are described at the *syntactical level*.

These sign expressions are implicitly defined through axioms (see 1.4). The remainder of terms and, given a different interpretation of formal systems, unspecified objects are introduced by explicit definitions.

Definition in logico-mathematical theories and formalised systems has a certain specificity.

Explicit definitions are continually used to extend the language of constructed theories or systems. It is therefore frequently the case that the introduction of a new sign expression through definition results in a transition from the system S_0 (where primitive terms appear, and possibly terms earlier introduced by definition) to the system S_1. The vocabulary of the system S_1 will contain as its correct part the vocabulary of the system S_0. One usually *intends* to reduce the new sign expressions (terms, designations), introduced by definition, to combinations of primitive terms according to definite rules (see 3.5). Each proposition of the language S_1, constructed out of sign expressions of the language S_0 and the sign expressions introduced into S_1, should be translatable into a proposition of the language S_0 and vice versa, the translation being performed on the basis of the logical means of the language S_1.

There are different interpretations of the concept of *translation* from one language into another. In this case, translatability of the propositions of the language S_1, containing the new term T^1 introduced by definition, into the propositions of the language S_0 lacking this term, means that we may replace T^1 by a combination of terms

belonging to S_0 in every proposition of S_1 containing T^1 according to the *rule* contained in the definition introducing T^1.

That means that in all extensions of S_0 to S_1, to S_2, etc., through successively introduced explicit definitions, the propositions containing T^1, T^2, etc., appearing in these extensions, may be successively eliminated and replaced by propositions of the language S_0. Moreover, if the proposition S_1 containing T^1 is deducible in the system S_1, it is also deducible in S_0.

For explicit definitions introduced into formalised theories and formal systems, it is important to emphasise their *abbreviative* function: they introduce a new term or sign instead of a complex combination of terms or signs.

Explicit definitions play a dual role in the constructions considered here: they function as *rules* extending the stock of transformation rules in the system S_0 and/or the role of the system's *theorems*. Thus, addition to the system S_0 of an explicit definition introducing a new term is regarded first and foremost as introduction of a new rule ensuring interchangeability of the Dfn and the Dfd. When the same definition is viewed as an element of the system S_1, it is frequently interpreted as a theorem of the system S_1, if the sign of equality by definition is replaced by the equality sign or the sign of logical equivalence. The justifiability of this substitution, of generation of the 'equality' or 'logical equivalence' relation by the relation of equality by definition (and consequently the justifiability of substitution of the Dfn for the Dfd and vice versa) will be considered in 3.5.

The equality-by-definition sign ($=\!\!=$) belongs to *metalanguage*,[1] when definitions are interpreted as rules for the interchangeability of the Dfn and Dfd. The proposition Dfd$=\!\!=$Dfn therefore also belongs to metalanguage propositions.

When definitions are interpreted as theorems, the sign$=\!\!=$ is correspondingly replaced by the equality sign ($=$) or that of logical equivalence ($\sim$) belonging already to the language of the system (when axioms and not axiom sche-

[1] Metalanguage, or metatheory, is a theory analysing the structure and methods of some other theory (language), the latter being called object language.

mata appear in the system). The proposition Dfd≡Dfn is regarded as a proposition of the system itself.

A definition introduced into the system S_0, the consistency of which has been proved, is consistent with it, if and only if its addition to the system S_0 does not result in a contradiction, that is, does not permit the deduction of two formulas in the system S_1 one of which is a negation of the other. If such is not the case, the definition proves to be inconsistent with the system S_0. Instances of this sort may occur when a new definition is introduced for some term T already defined in the system.

In what way are explicit definitions interpreted in formal systems and formalised theories?

An explicit definition in formal systems and formalised languages is interpreted as a proposition satisfying the following conditions:

a) it introduces a new term[1] as an abbreviation for a combination of terms obtained on the basis of the rules for the system S_0;

b) the new term introduced by this definition belongs to the S_1 but does not belong to S_0;

c) in a certain sense it extends the stock of rules for the transformation of S_1 as compared with the stock of rules in the system S_0; the stock of correctly constructed expressions of the system S_1 and the number of theorems provable in it are also extended;

d) it is assumed, however, that all the new propositions obtained by extension of S_0 and appearing only in S_1 may be translated into propositions of the language S_0. Extension of S_0 to S_1 is thus inessential.

Depending on whether we consider definitions in formalised interpreted theories and systems or in uninterpreted ones, definitions may be divided into meaningful semantic ones and formal semantic ones. In formalised interpreted theories (e.g., propositional logic) logical copulas are defined by meaningful semantic definitions.

In handling formal systems or ignoring interpretations of terms appearing in definitions, we deal with formal syntactical definitions. Definitions are in this case written in the form *Dfd*≡*Dfn*, where the sign ≡ means that the

[1] In this chapter, 'term' is taken to mean a sign expression in general.

Dfd is an abbreviated notation of the Dfn and that the Dfd and the Dfn are mutually translatable, mutually interchangeable in various contexts.

Having introduced a new term into the language of a theory by a meaningful semantic definition ($\mathrm{Dfd} =_{Df} \mathrm{Dfn}$), we sometimes begin to consider the Dfd to be a reduced notation for the *Dfn* (*Dfd* $\rightleftarrows$ *Dfn*). Thus, introducing the equivalence sign $\sim$ in Boolean algebra by the meaningful semantic definition

$$p \sim q =_{Df} (p \to q) \wedge (q \to p)$$

and substantiating it by the fact that one and the same Boolean function corresponds to the expressions to the left and to the right of the sign $=_{Df}$, we can then pass on to the level of formal manipulation of the formulas and write this definition like this:

$$p \sim q \rightleftarrows (p \to q) \wedge (q \to p).$$

In a meaningful semantic definition the left and right parts were identified with reference to the objects (the Boolean functions), which ensures interchangeability of the Dfd and the Dfn. On this basis, the transition is effected to the corresponding formal syntactical definition, where the Dfd is considered to be an abbreviation of the Dfn so that the Dfd and Dfn become interchangeable and mutually translatable.

To reduce complex descriptions, simple expressions including the sign $\overline{\rightleftarrows}$ (*Dfd* $\overline{\rightleftarrows}$ *Dfn*) are sometimes introduced for temporary convenience.

Thus the expression $A \overline{\rightleftarrows} P(x_1, x_2, \ldots, x_n) \vee \overline{\wedge} P(x_1, x_2, \ldots, x_n)$ means that to reduce its right part one may write A, and if the need arises, one may write instead of A the expression to the right of the sign $\overline{\rightleftarrows}$.

Expressions with the sign $\overline{\rightleftarrows}$ occurring in formal systems may be regarded as auxiliary definitions. As distinct from other definitions, they do not extend the language of a system: the sign A in the above expression is not a component of the language system. The sign $\overline{\rightleftarrows}$ is sometimes regarded as the sign of graphic equivalence: the abbreviating notation A is introduced to denote the notation which is abbreviated. Definitions with the sign $\overline{\rightleftarrows}$ are analogous to the semantic definitions considered above (see 1.2). However, while the terms introduced by semantic definitions are included in

the language of the system, the sign expressions introduced by the sign $\overline{\circ}$ are auxiliary signs of the system and are usually eliminated in the final results of proofs and discourses.

As pointed out earlier, definitions in logico-mathematical theories and systems may function both as rules and as theorems (the latter, however, may be used as corresponding rules, too). When an axiomatic system is constructed, for instance, on the basis of a minimal set of mutually indefinable logical copulas, some logical copulas may be introduced by definitions functioning as rules in this system.

Thus the primitive terms of J. Lukasiewicz's system constructed as a propositional calculus include the signs for implication and negation described in the following axioms:

(1) $(p \to q) \to ((q \to z) \to (p \to z))$,
(2) $(\overline{\ } p \to p) \to p$,
(3) $p \to (\overline{\ } p \to q)$.

The system also contains the separation rule and the substitution rule supplemented by definitions of the following logical copulas:

(1) $\varphi \vee \psi \rightleftarrows \overline{\ } \varphi \to \psi$,
(2) $\varphi \wedge \psi \rightleftarrows \overline{\ } (\overline{\ } \varphi \vee \overline{\ } \psi)$,
(3) $\varphi \sim \psi \rightleftarrows (\varphi \to \psi) \wedge (\psi \to \varphi)$.

These definitions are not axioms and theorems of the system: they are parts of the rule for substituting entities equal by definition (extensionality rule).

The general formulation of this rule is this:

$$\frac{\varphi \rightleftarrows \psi \qquad X}{X\,(\varphi \| \psi)}.$$

The substitution rule by definition permits addition of the expression X $(\varphi\|\psi)$ to the system by substituting the φ part for some part in the theorem coinciding with ψ. The signs φ and ψ are metalanguage symbols that may be replaced by any formulas of the system.

In applied systems (which usually contain special predicate or functional symbols) based on the narrow predicate calculus with equality, definitions are usually interpreted

as theorems of the system. In such cases, the *primary function* of definitions is that of introducing definitions among the theorems of the system.

Here we shall consider three kinds of such definitions, namely, normal, conditional, and descriptive.[1] Inductive and recursive definitions as implicit definitions, as well as the problem of reducing them to explicit ones will be considered later (see 3.5).

3.2. NORMAL DEFINITIONS AND ADDITIONAL DEMANDS ON THEM

Normal definitions have the form of equivalences or equalities. Definitions introduced into the system with the aid of the signs $=_{Df}$ or $\rightleftharpoons$ (which in this case play the role of secondary postulates of the system analogous to axioms) may later be regarded as theorems of the system, where the left and the right parts are connected by the signs $\sim$ (equivalence) or $=$ (equality). Thus, if the left and the right parts of definitions are certain propositional functions, definitions with the sign $=_{Df}$ are transformed into theorems with the sign $\sim$.

Assume that a definition is used to specify (not introduce) the operation of subtraction:

$$x-y=z=_{Df} z+y=x,$$

where the variables range over real numbers. Then this definition may be written as a theorem in the form: $x-y=z\sim z+y=x$, which should be read as follows: 'A ternary of numbers will satisfy the function $x-y=z$, if and only if it satisfies the function $z+y=x$.'

Using the substitution rule, we may obtain from this theorem certain results for particular values of x, y, and z.

If definitions are introduced in logico-mathematical systems with the aid of the sign $\rightleftharpoons$, they contain, first and foremost, the rule for the interchangeability of the Dfd and the Dfn in any contexts. However, they may then be formulated as theorems containing the sign $\sim$ (that is, the equivalence sign). For instance, the definition

[1] See J. Słupecki, L. Borkowski, *Op. cit.*, pp. 140-48.

'The prime number $(x) \rightleftharpoons x>1 \wedge \forall y\,(y \neq 1 \wedge y/x \rightarrow y=x)$' may be represented as a proposition with the equivalence sign:

'The prime number $(x) \sim x>1 \wedge \forall y\,(y \neq 1 \wedge y/x \rightarrow y=x)$', and it is true (given a corresponding interpretation) and provable in theories based on the restricted predicate calculus with equality.

That is why definitions in systems with equality and logical equivalence are often introduced at the level of object language directly as corresponding equalities or equivalences (Dfd=Dfn, Dfd$\sim$Dfn). They are interpreted as theorems containing at the same time the permission to substitute equal entities for each other in other propositions of the system.

Since the theorem of deduction is valid in the restricted predicate calculus with equality and since equivalence is also introduced in it, the definitions Dfd$\rightleftharpoons$Dfn introducing new sign expressions in the corresponding logico-mathematical systems (e.g., in certain fragments of formalised arithmetic of natural numbers) and primarily interpreted as the rules for substituting an equal entity for an equal entity, may always be regarded as theorems of the form Dfd$\sim$Dfn.[1]

Assume that we are introducing by definition a certain function φ including a new constant predicate P which appears in the arithmetic of natural numbers as an abbreviation for the function ψ that includes other predicates (see the previous example of the definition of prime numbers). The definition $\varphi \rightleftharpoons \psi$ may then be correlated with the expression $\varphi \sim \psi$ which will be a theorem of the system (that is, $\vdash \varphi \sim \psi$). Indeed, if we add φ to the set of axioms and earlier proved propositions (theorems) of the system (let us call it the π-set) we can obtain ψ according to the rule contained in the definition that we accepted ($\varphi \rightleftharpoons \psi$). And vice versa, we may obtain the formula φ by adding the formula ψ to π. And that means that the formulas $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$ will be proved according to the theorem of deduction. Consequently, the formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ will also be proved. And that is, by definition, the formula $\varphi \sim \psi$. Consequently, $\varphi \sim \psi$ is a theorem.

[1] We assume that in the expression Dfd $\rightleftharpoons$ Dfn not only may the Dfd be substituted for the Dfn, the former being an abbreviation of the latter, but the Dfn for the Dfd as well, which, properly speaking, requires special substantiation (see 3.5).

Indeed,

$$\frac{\pi,\ \varphi \vdash \psi}{\pi \vdash (\varphi \to \psi)}; \qquad \frac{\pi,\ \psi \vdash \varphi}{\pi \vdash (\psi \to \varphi)}.$$

If the formulas $\varphi \to \psi$ and $\psi \to \varphi$ are deducible from π, the formulas $\pi \to (\varphi \to \psi)$, $\pi \to (\psi \to \varphi)$ will also be proved (according to the theorem of deduction). But since our π comprises only the theorems (axioms included), and may be viewed as a conjunction of the latter, the formulas $\varphi \to \psi$, $\psi \to \varphi$ will also be proved according to the *modus ponens* rule, that is, the theorem $\varphi \sim \psi$ will be proved. We usually apply the rule for introducing definitions in systems that are based on predicate calculus. 'There is no need to introduce this rule within the predicate calculus itself as a new basic rule, since the rules listed above suffice to prove all the true expressions of this system.'[1]

In the metalanguages of the various logical calculi, however (where the terms Dfd and Dfn, not just the sign $=$, are metasigns), we frequently resort to definitions as a means for introducing new terms and corresponding concepts (e.g., the concepts of canonical forms for formulas, the concepts of completeness of calculi, their consistency, independence, demonstrability, definability, etc.).

Let us cite some examples of normal definitions:

(1) The prime number $(x) \sim x > 1 \wedge \forall y\,(y \neq 1 \wedge y/x \to y = x)$,

(2) $x \leqslant y \sim \overline{\forall}\,(x > y)$,

(3) $X \subset Y \sim \forall x\,(x \in X \to x \in Y)$,

(4) $\mathrm{Seq}\,(n) = n + 1$,

(5) $x - y = z \sim z + y = x$.

Definition (1) defines a one-place predicate, definitions (2) and (3), two-place predicates ('less or equal', the relation of including the set X in the set Y), definition (4), the one-place name functor ('indicator-function') 'following n', and definition (5), a two-place name functor, viz, the operation of subtraction. In definition (5) the Dfd is a certain object, namely, the operation of subtraction, and the

[1] J. Słupecki, L. Borkowski. *Op. cit.*, p. 148; see also pp. 146-47.

Dfn, a propositional function describing the conditions under which the subtraction operation is performed. Assuming this operation has been defined in some way before definition (5) is introduced, the expressions to the left and to the right of the sign $\sim$ may be viewed as propositional functions (see an interpretation of this example on p. 113).

Apart from the rules listed in the previous chapter normal definitions must satisfy a number of other requirements.

(1) *Each variable should occur in the Dfd only once.* Unless this condition is satisfied, it may prove to be impossible to eliminate the Dfd in some contexts. Assume that we define logarithm in the following way: $\log_a a=b\sim a^b=a$. For example, in a context like $\log_{10} 10=1$ the term *log* may be eliminated according to the rule contained in our definition, being replaced by the expression $10^1=10$. But in a different context, e.g., $\log_{10} 100=2$, the term *log* cannot be eliminated in accordance with the rule contained in the definition given above since 10 and 100 are not values of one and the same variable *a*.

In the mathematical formulas of the equality type where all the variables are thought of as connected by universal quantifiers, these variables in the formula must take one and the same value in all the acts of substituting the concrete values of the corresponding domain for each variable (owing to the elimination of the universal quantifiers). If the formula contains *different* variables, the latter may also take *one and the same value* when replaced by the corresponding values. If the formula is true for any value of the domain (e.g., for the domain of the natural numbers), it is also true for the particular case where its values for the variables are the same. Thus, the formula $(a+b)^2=a^2+2ab+b^2$ is also true for the case where *a* and *b* take identical values. But this formula cannot be used in such a way, for instance, that in eliminating the universal quantifiers *a* take the value 5 in the right part of the formula and 10 in the left part.

(2) *The requirement of homogeneity: a free variable in the right or left part of the definition should also be free in its left or right part, respectively.*

This requirement was already formulated concisely in the previous chapter (see 2.3) as part of the rule of partial identity of the structures of the Dfd and Dfn terms. If it is not satisfied, a contradiction may arise.

Assume that we are given the definition of parallelism: '$a \| b \equiv$ the straight lines a and b lie in the plane α and have no common point.' Let us formulate this definition as a proposition with the sign $\sim$ (the equivalence sign): '$a \| b \sim$ the straight lines a and b lie in the plane α and have no common point.' Let a_1 and b_1 be some concrete straight lines which lie in some plane α_1 and have no common point. Substituting a_1 for a, b_1 for b, α_1 for α in our definition, we shall obtain: '$a_1 \| b_1 \sim$ the straight lines a_1 and b_1 lie in the plane α_1 and have no common point.' The right part of the equivalence is true (by condition). Then, using the *modus ponens* rule for equivalence right to left, we shall obtain: $a_1 \| b_1$ is true.

Substituting now a_1 for a, b_1 for b, α_2 for α, we shall obtain: '$a_1 \| b_1 \sim$ the straight lines a_1 and b_1 lie in the plane α_2 and have no common point.' The straight lines a_1 and b_1 cannot lie in any plane other than α_1. The first part of the conjunction in the expression to the right of the sign $\sim$ is therefore false. It follows that the whole of the conjunction 'the straight lines a_1 and b_1 lie in the plane α_2 and have no common point' is false. Let us now negate both parts of the equivalence. We shall obtain: '$\neg(a_1 \| b_1) \sim \neg$ (a_1 and b_1 lie in the plane α_2 and have no common point).' The right part of the equivalence is true. Application of the *modus ponens* rule right to left will yield the result that $\neg(a_1 \| b_1)$ is also true. Thus correct substitutions in the definition have yielded a contradiction $(a_1 \| b_1) \wedge \neg(a_1 \| b_1)$. If, however, the variable α occurred in both parts of the definition reading '$a \| b$ in the plane $\alpha \sim$ the straight lines a and b lie in the plane α and have no common point', this contradiction would not arise.

(3) *The requirement of existence and singularity.* The Dfd in definition (5) includes the equality relation $x - y = z \sim$ $\sim z + y = x$, the expression $x - y$ being introduced into the system for the first time as a certain object function and not as a predicate. The general form of this sort of normal definitions is as follows:

$$f(x_1, x_2, x_3 \ldots, x_n) = y \sim F(x_1, x_2, \ldots, x_n, y).$$

where f is a name functor of n arguments, and the Dfn — a propositional function in which $x_1, x_2, x_3, \ldots, x_n, y$ are free variables.

Apart from the requirements (1) and (2), normal definitions of this form must satisfy the requirements of existence and singularity. This requirement is satisfied if and only if the formula

$$\forall x_1 \forall x_2 \forall x_3 \ldots \forall x_n \exists! y F(x_1, x_2, x_3, \ldots, x_n, y)$$

is a theorem of the corresponding system (for every x_1, x_2, x_3, . . ., x_n there must be a y, and only one y, such that it satisfies the function F).

If this formula is not true, if it is not a theorem, the definition of the form $f(x_1, x_2, x_3, \ldots, x_n) = y \sim F(x_1, x_2, x_3, \ldots, x_n, y)$ may result in a contradiction.

In example (5) cited above the role of y is played by z, and the definition satisfies the requirements of singularity and existence.

The formula

$$\forall x_1 \forall x_2 \forall x_3, \ldots, \forall x_n \exists! y F(x_1, x_2, x_3, \ldots, x_n, y)$$

is here a theorem. If the object domain comprises real numbers, then for any x and y there exists a single z such that $z+y=x$.

Let us consider one more example. Assume that we have a definition of the operation of division:

$$\frac{x}{y} = z \sim y \cdot z = x.$$

If we now substitute 0 for x and y and first 1 and then 2 for z, we shall respectively obtain true propositions $0 \cdot 1 = 0$; $0 \cdot 2 = 0$ in the right-hand part of the equivalence. Applying the *modus ponens* rule right to left, we shall obtain respectively the proposition: $0 : 0 = 1$, $0 : 0 = 2$.

Since left-hand parts in the equalities obtained are one and the same object $(0 : 0)$, the right-hand parts are also one and the same object, that is $1 = 2$. But we know that $1 \neq 2$. We have thus come to a contradiction precisely because for some values of x and y, z is not the only entity satisfying the function $y \cdot z = x$.

3.3. CONDITIONAL DEFINITIONS AND DEFINITIONS WITH THE ι-OPERATOR

Conditional definitions are definitions having the form of implications, where the consequent is a formulation

of a normal definition with equality of the type

$$f(x_1, x_2, x_3, \ldots, x_n) = y \sim F(x_1, x_2, x_3, \ldots, x_n, y),$$

and the antecedent is a formulation of the condition that must be satisfied if existence and singularity of y are to be attained.

In considering normal definitions of the form $f(x_1, x_2, x_3, \ldots, x_n) = y \sim F(x_1, x_2, x_3, \ldots, x_n, y)$ we indicated that these definitions must satisfy the requirement of existence and singularity, that is, for every $x_1, x_2, x_3, \ldots, x_n$ there must be a single y which satisfies the predicate F. Failure to comply with this requirement, as we have seen, may result in a contradiction. In conditional definitions these requirements are formulated in concrete form in the antecedent of the definition.

Thus the normal definition

$$\frac{x}{y} = z \sim y \cdot z = x$$

may be formulated as a conditional one:

$$y \neq 0 \rightarrow \left(\frac{x}{y} = z \sim y \cdot z = x\right).$$

This definition formulates the condition of existence and singularity. In the absence of this condition, that is, when $y=0$, various z will satisfy the correlation $y \cdot z = x$ in our definition when $x=0$ and $y=0$ (cf. $0 \cdot 1 = 0$; $0 \cdot 2 = 0$; $0 \cdot 3 = 0$, etc.). That means that the theorem

$$\forall x \forall y \exists! z (y \cdot z = x).$$

does not obtain. If the condition of existence and singularity is satisfied (and it is satisfied when $y \neq 0$), the formula

$$\forall x \forall y \exists! z (y \cdot z = x)$$

is a theorem.

The conditional definition may be written in its general form as

$$\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \rightarrow [f(x_1, x_2, \ldots, x_n) = y \sim F(x_1, x_2, \ldots, x_n, y)],$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ are all or some variables $x_1, x_2, \ldots, x_n$, $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ is a propositional function,

and the expression to the right of the implication sign is a normal definition.

Let us consider another example of a conditional definition:

$$y \neq 0 \wedge u \neq 0 \rightarrow \left(\frac{x}{y} = \frac{z}{u} \sim x \cdot u = y \cdot z\right).$$

x_{i_1} and x_{i_2} (that is, the variables y and u) constitute here a correct subset of the variables $x_1, x_2, \ldots, x_n$ appearing in the right-hand part of the implication. The definition of proportion is correct when the limitations formulated in the condition of this definition are satisfied.

To introduce new objects into a system, definitions are frequently used in which the Dfn contains an ι-operator. This operator is usually employed to form the so-called individual descriptions, that is, expressions of the form '$\iota x(\ldots x \ldots)$' and the corresponding individual objects or descriptums ('that individual for which $\ldots x \ldots$'), e.g., 'that individual x which has the property P, that is, satisfies the function $P(x)$'. If a description satisfies the condition of singularity and existence, that is, if it is used to single out the only object of the domain, the description $\iota x(\ldots x \ldots)$, that is, a certain descriptive name, may be replaced by a new abbreviated name by means of a definition. In cases like this, one speaks of definitions through descriptions (for a formal definition of definite description see 1.6; descriptive definitions are discussed on a meaningful level in 1.5).

3.4. INDUCTIVE AND RECURSIVE DEFINITIONS

Following S. C. Kleene,[1] we shall distinguish between inductive and recursive definitions. *Inductive definitions are those which permit to construct new objects of a theory from certain primitive objects of the theory by the means of certain operations and in such a way that no other sequences of primitive symbols of the theory (except objects constructed in this way) can be objects of this theory.* Inductive definitions are used most frequently in arithmetic and logic.

Let us consider some examples.

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[1] See S. C. Kleene, *Op. cit.*, p. 217.

(a) *An inductive definition of natural numbers.*

The primitive objects here are 0 and the operation 'following n', that is, an operation permitting the transition from n to $n+1$. It is marked by the sign $'$ ('n'' means 'following n').

The definition consists of a number of points: (1) 0 is a natural number; (2) if n is a natural number, then n' is also a natural number; (3) there are no natural numbers other than those obtained by (1) and (2).

As this definition does not express explicitly that the numbers generated by different modes of applying (1) and (2) will always be different, this definition of natural numbers is usually supplemented by two further points: (4) for any numbers n and m, if $m'=n'$, then $m=n$; in a similar way, for any natural numbers m and n, if $m=n$, then $m'=n'$; (5) for any natural number n, $n'\neq 0$. The introduction of points (4) and (5) in the definition is due to the fact that the operation 'following n' is a single-valued function or univalent operator: these points guarantee that any two numbers will be different if they have been generated in different ways.

(b) *An inductive definition of a formula in propositional calculus.*

The primitive objects here are the propositional variables (p), (q), (r), (p_1), (q_1), (r_1), (p_2), . . ., and the basic operations $\neg$, $\wedge$, $\vee$, $\rightarrow$ (the operations of negation, conjunction, disjunction, implication).

The definition consists of the following three points:

(1) Every propositional variable (p), (q), (r), (p_1), . . ., is a formula.

(2) If (φ) and (ψ) are formulas, then $\neg(\varphi)$, $((\varphi)\wedge(\psi))$, $((\varphi)\vee(\psi))$, $((\varphi)\rightarrow(\psi))$ are also formulas.

(3) The calculus does not have any formulas other than those generated by the application of (1) and (2).

(c) *Definition of even number.*

The primitive object here is the number 0, the basic operation, that of adding 2 ($+2$). The definition consists of the following points:

(1) 0 is an even number.

(2) If n is an even number, then $n+2$ is an even number.

(3) There are no (natural) even numbers other than those generated by (1) and (2).

(d) *The definition of a provable formula (theorem) in axiomatic propositional calculi containing rules for separation and substitution.*

The primitive objects here are the axioms of a system. The basic operations are the separation rule and the substitution rule.

The definition consists of the following points:

(1) Every axiom is a provable formula.

(2) Every formula obtained from the axioms by the application of the separation and substitution rules (as well as those obtained in the same way from earlier proved formulas) is a provable formula.

(3) There are no formulas other than those generated by the application of points (1) and (2).

Two basic types of inductive definition are distinguished, *fundamental and nonfundamental ones. Fundamental definitions are those which generate some basic domain from primitive objects by basic operations applied to them. Nonfundamental definitions are those which single out a certain subset in a previously defined domain.*

The definition (a) of a natural number cited above is a fundamental inductive definition as it generates a definite domain to be studied (natural numbers). Of the same nature is the definition (b) of a formula in propositional calculus. On the contrary, definition (c) of even number is nonfundamental: it is assumed that the domain of natural numbers is given from the outset or generated by fundamental definition, and inductive procedures are used to define a certain subset of the set of natural numbers, namely, even numbers. In defining this subset, we simultaneously define the predicate 'to be an even number'. Definition (d), that is, the definition of a provable formula, is also nonfundamental: it is assumed that the domain of the calculus formula is already given or generated in its turn by inductive definition, and we are using inductive procedures' to construct its subset. The predicate 'to be a provable formula' corresponds to this subset.

It is sometimes said that nonfundamental inductive definitions are used to define a certain predicate, e.g., $P(x)$; the range of values of the variable x coincides with the object domain.

Thus the fundamental inductive definition establishes the range of a variable, over which one may subsequently

define predicates by nonfundamental inductive definitions [1] Whether an inductive definition is fundamental or nonfundamental becomes clear from an analysis of the corresponding context: it depends on the nature of the theory that resorts to this sort of definition. Thus, the fundamental definition of natural number may be viewed as a nonfundamental one if natural (whole and positive) numbers are considered as part of some other domain, e.g., as a subset of whole positive and negative numbers.

A distinction is made between *direct and oblique points* in inductive definitions. The direct points of a fundamental definition determine the construction of a domain. The direct points of a nonfundamental definition also describe the modes of construction of the objects of a certain subset of the corresponding domain whose elements satisfy the predicate corresponding to this subset (e.g., make the predicate $P(x)$ true). The oblique points of fundamental and nonfundamental definitions assert that there are no objects distinct from those indicated in the direct points of the given definition.

In definition (a) points (1) and (2) are direct, point (3), oblique; in a similar way, in definitions (b), (c), (d), points (1) and (2) are direct and point (3), oblique.

Direct points are divided into *basic* and *inductive*. Basic points are those in which examples are given of the objects to be defined. Inductive points are those which describe the procedure for the generation of new objects from the primitive objects. In all the examples of definitions point (1) is basic and point (2), inductive.

The order of generating objects by means of fundamental inductive definitions is strict (which is not always the case in generating objects by means of nonfundamental inductive definitions). That means that the objects of the generated domain differ from one another if they have been generated by different inductive procedures or application of a different number of steps of one and the same procedure.

Definition (a) formulates points (4) and (5) substantiating just this difference between two differently generated objects. Proceeding from these points, it is easy to prove, e.g., that $0'''\neq 0''$.

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[1] S. C. Kleene. *Op. cit.*, p. 259.

Let $0''''$ be the value of m', and $0''$, the value for n' in points (4) and (5). Assume that $0''''=0''$. By point (4), these objects may be equal if $0'''=0'$ (that is, if the preceding objects are equal). By applying the same point, we obtain $0''=0$. But this cannot be by virtue of point (5). Since an object following n (i.e., n') cannot be equal to zero, our assumption is wrong. Consequently, $0''''\neq 0''$.

If natural numbers are viewed as generated by a fundamental inductive definition, we may substantiate application of the principle of complete mathematical induction for proving some statements about the natural series of numbers.[1] This principle may be formulated as follows: assume that P is some property of natural numbers. Let it be that

(1) 0 has the property P;

(2) if some natural number n has the property P then its successor n' has the property P.

It follows that any n has the property P.

Assume that there is a property P for which points (1) and (2) hold. If we have an inductive definition of an arbitrary number n, then by generating it under points (1) and (2) of our definition (a) and starting with 0, we can verify (in actual fact or in principle by proceeding from the abstraction of potential realisation), whether it has the property P or not.

In other words, points (1) and (2) are tools which enable us, while we are generating the natural numbers by points (1) and (2) of the inductive definition to verify for each number as we generate it whether it has the property P.[2]

There are disagreements between representatives of the classical trend in logic and mathematics and those scholars who adhere to the principles of constructivism (A. Markov, P. Lorenzen, H. Curry and others) over inductive definitions. The fact is that inductive definitions include indirect indications that sets of objects (e.g., of natural numbers) are here defined in such a way that for each element it may be decided from the procedure of generating them ('the enumerative function') whether or not it belongs to the set. That means that definitions with an oblique point satisfy

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[1] See S. C. Kleene, *Op. cit.*, p. 21.

[2] *Ibid.*

the predicate of 'membership in a given set (box)' with regard to each element of the set.

The predicate of 'membership of an object in a given box' assumes the ability to determine whether every object of the set is constructed from the given primitive object by given operations or not. The possibility of satisfying this predicate for each element of the set defined inductively is based on the oblique point: there are no objects belonging to our set apart from those that have been constructed from certain primitive objects in a certain way. Every object of a domain or set therefore either possesses the given property or does not possess it, it either belongs to the box or it does not.

Already the most general principles of constructivism and intuitivism make it clear that the oblique point of inductive definition is unacceptable for constructivists. Hence their critique of the applicability of the law of excluded middle in such cases.

Indeed, if membership in the given box is taken to mean the construction of the object from the primitive ones with the aid of given operations, what does the statement mean that every object of our set belongs or does not belong to the box? It stands to reason that inability to construct a certain object does not yet mean that it cannot be constructed in the generation of the entire infinite set of objects. Moreover, if an object cannot be constructed in a given way, it may be constructed in some other way and it will therefore belong to our box. This problem always arises as a nontrivial one when there are more than one primitive object and more than one generative operation (cf. the inductive definition of provable formula above).

Representatives of the classical trend in mathematics and logic interpret the nonexistence of an object (its nonmembership in the given box) in the following way: an object does not belong in the given box if and only if the assumption of its existence, its membership in the given box, results in a contradiction.

The opponents of the classical trend point out, however, that the propositions 'An object belongs in the given box' (in the sense of its direct generation) and 'An object does not belong in the given box' are not logical negations of each other for the following reasons. The proposition 'An object belongs in the given box' (in the sense of its effective construction or retrieval) records a certain experimental

fact, while the proposition 'An object does not belong in the given box' (in the sense of logical reasoning resulting in absurdity) is not an experimental fact. We cannot therefore, assert that one of these propositions is necessarily true. The law of excluded middle proves to be inapplicable here.

It has been indicated that objects constructed with the aid of fundamental definitions are considered different from one another if they are generated by different operations (or if there is a difference in the number of applications of one and the same operation). These conditions for distinguishability of objects constructed in different ways are precisely what is registered in points (4) and (5) of the definition of natural number. *Objects generated in different ways in nonfundamental definitions are not necessarily different.* For instance, one and the same provable formula may be obtained in calculi by different operations and in different ways.

In nonfundamental definitions a different principle of identifying objects is used. Here we consider the objects to be identical, though not everything that may be stated about one object may be stated about another: although the methods for constructing these objects may differ, we nevertheless regard the objects as identical.

Modern constructivists (logicians and mathematicians) thoroughly criticise inductive definitions in their classical interpretation. They hold that inductive definitions should include only the first two points. The reason for this is as follows. Fundamental inductive definitions are not used when objects of a certain set are introduced; they are used to introduce the objects of a certain domain that is not finite. The sets are then introduced by predicates or inductive nonfundamental definitions with regard to an already generated domain by a fundamental inductive definition. The objects in a domain generated by inductive definitions are constructivist in character. The abstraction of actual infinity is excluded here, but the abstraction of potential realisability is retained, serving as a frame of reference for the generated objects of the domain.

If this approach is taken, the oblique point (point (3) of the definition of natural number given above) turns out to be simply meaningless as it assumes all the objects to be in some sense given beforehand. Points (4) and (5) of the same

definition also prove to be superfluous. In these points the axioms serve to define the conditions of identity and difference for the generated objects: they are different when they are constructed in different ways. These conditions, however, do not refer to the inductive *fundamental* definition proper. It is clear from the process of constructive generation of objects itself that objects constructed from the primitive object in different ways (01, 011, 0111, etc.) are different and those constructed in a similar way, identical (e.g., 01 and 01). We resort here to the identification abstraction.

These conditions may be explicitly formalised as follows.

(1) $0=0$ (the primitive object equals itself);

(2) for any objects m and n, if $m=n$, then $m'=n'$.

This definition is *nonfundamental*. It may be used to identify and distinguish the objects of a constructively generated domain.

Inductive definitions are frequently regarded as purely formal (uninterpreted ones). The expressions (words) generated in such cases are therefore material objects in their own right. These objects are later given an interpretation, a correlation being established between them and other objects (sometimes also material, and sometimes abstract).

Syntactical systems of this sort do not free us from epistemological problems. Complicated problems involving abstraction and the formation of abstract objects arise, for example, when we begin to interpret generated objects as numbers perceived as certain sets. Kleene says with some justification that in inductive definitions we disengage ourselves from the inner nature of the objects defined. Every individual defined is regarded only as an object occupying a concrete position in the natural series, a natural number being regarded as given if its generation according to the inductive definition is given.

When we deal with fundamental inductive definitions generating a domain whose objects constructed by different inductive procedures are different, the definitions are a necessary premise for using the so-called *recursive definitions*, or *definitions by induction*. Recursive definitions are used to define functions and predicates over subsets of an inductively generated domain. If the objects of subsets defined by nonfundamental inductive definitions are constructed in strict order (or an order that may be transformed into strict order), the corresponding functions and predicates may also be

defined over these subsets. When such is not the case, this cannot be done. Assume that we have a nonfundamental definition of a provable formula. The order of generating provable formulas is not strict: formulas generated in different ways are not always different. Assume that we recursively define the function $\varphi(A)$ over a certain set—the function of being the length of the derivation of the formula A in a calculus with a substitution rule and the *modus ponens*. We shall not obtain in this case a nonambiguous definition of $\varphi(A)$, as the various formulas A may be generated by the same number of steps.

Recursive definitions are methods for defining the arithmetical function $\varphi(y)$ or the predicate $P(y)$ through the domains of the value of these functions or the predicates for the values of their arguments.

These definitions explicitly indicate certain values of the functions defined and describe the means for obtaining new values from the already defined ones through substitutions and various 'recursive schemes'. Thus, we may define the values of $\varphi(0)$ or $P(0)$. We may then express the values of $\varphi(y')$ or $P(y')$ in terms of y and $\varphi(y)$ or y and $P(y)$ for every y. Under these circumstances, the value of $\varphi(y)$ or $P(y)$ is defined for every y.

Let us cite an example.

$$\begin{cases} \varphi(0) = q, \\ \varphi(y') = X(y, \varphi(y)). \end{cases}$$

Assume that $\varphi(0) = 3$ and X is the function of addition, then

$$\begin{cases} \varphi(0) = 3, \\ \varphi(y') = y + \varphi(y). \end{cases}$$

Proceeding from these data, we can calculate the values of φ for 1, 2, 3, 4, etc. Let us calculate the value of φ for 1:

$$\varphi(1) = \varphi(0') = 0 + \varphi(0) = 0 + 3 = 3$$

(this calculation is performed according to the formula $\varphi(y') = y + \varphi(y)$, where 0 is substituted for y and the value 3 is substituted for $\varphi(0)$, according to the equality $\varphi(0) = 3$ of our definition).

Let us calculate the value of the function $\varphi(2)$:

$$\varphi(2) = \varphi(1') = 1 + \varphi(1) = 1 + 3 = 4$$

(3 may be substituted for $\varphi(1)$ since $\varphi(1)$ has already been calculated).

Let us calculate the value of $\varphi(3)$:

$$\varphi(3) = \varphi(2') = 2 + \varphi(2) = 2 + 4 = 6.$$

Let us calculate the value of $\varphi(4)$:

$$\varphi(4) = \varphi(3') = 3 + \varphi(3) = 3 + 6 = 9, \text{ etc.}$$

Thus the recursive definition (or the inductive definition)

$$\begin{cases} \varphi(0) = 3, \\ \varphi(y') = y + \varphi(y) \end{cases}$$

serves to obtain a series of the natural numbers: 3, 4, 6, 9, . . ., constituting a set of the values of the function defining it in an explicit manner. An implicit definition is transformed into an explicit, tabular one by applying certain procedures. However, since the set of values for the function φ is infinite, this definition of the function through a finite set of its values would be incomplete if we did not supplement it by a rule for calculating the values for any of its arguments. These rules, however, are formulated implicitly, in the form of the recursive definition cited above.

In other recursive definitions the function defined may depend on additional parameters which take fixed values in the induction on y.

A recursive definition of the addition function is formulated in general form as

$$\begin{cases} a + 0 = a, & (1) \\ a + b' = (a + b)'. & (2) \end{cases}$$

Let us consider the example of calculating the sum of $5+2$ on the basis of a recursive definition of addition. Let a be the parameter whose value is fixed and equals 5. In this example, just as in the previous one, it is assumed that we know in what way different numbers may be represented by the operation 'following n', e.g., $2' = 3$, $0' = 1$; $3 = 1''$, etc.

(a) $5 + 2 = 5 + 1' = (5 + 1)'$, | (by equation 2)

(b) $5 + 1 = 5 + 0' = (5 + 0)'$, | (by equation 1)

(c) $5 + 0 = 5$.

Let us now go back from equation (c).

Since $5+0=5$, then $(5+0)'=5'=6$, therefore $5+1=6$, too (see equation (b)).

Since $5+1=6$, then $(5+1)'=6'=7$, therefore $5+2=7$ (see equation (a)).

Thus $5+2=7$.

Computability of arithmetical functions defined recursively is based on a class of some other functions believed to be given and known from the start. Operations corresponding to these functions may serve as the basis for calculating, in an elementary fashion, the values of more complex functions defined recursively.

Thus, if this class of previously given functions consists of the following five functions (recursive schemes)

(1) $\varphi(x) = x'$,

(2) $\varphi(x_1, \ldots, x_n) = q$,

(3) $\varphi(x_1, \ldots, x_n) = x_i$,

(4) $\varphi(x_1, \ldots, x_n) = \psi[X_1(x_1, \ldots, x_n), \ldots, X_m(x_1, \ldots, x_n)]$,

(5a) $\begin{cases} \varphi(0) = q, \\ \varphi(y') = X(y, \varphi(y)), \end{cases}$

(5b) $\begin{cases} \varphi(0, x_2, \ldots, x_n) = \psi(x_2, \ldots, x_n), \\ \varphi(y', x_2, \ldots, x_n) = X(\varphi(y, x_2, \ldots, x_n), x_2, \ldots, x_n)^1, \end{cases}$

then the functions whose meanings may be defined through operations corresponding to these primitive functions (including the functions (1)-(5) themselves), may be called *primitive recursive.*

Function (1) correlates a certain object of a series (sequence) with an object following it. Function (2) correlates the values $x_1, \ldots, x_n$ with some natural number q. Function (3) correlates the values $x_1, \ldots, x_n$ with one of them, etc. If recursive functions are defined on series of objects which do not satisfy the conditions formulated in defining natural number, they may prove to be *contradictory.* Let us quote an example.[2]

Let us consider two sets of objects M and N. The set M consists of *different* objects α and β. The set N consists of *different* objects 1, 2 and 3. Let the elements of these sets

[1] See S. C. Kleene. *Op. cit.*, p. 219.

[2] R. Dedekind. *Was sind und was sollen die Zahlen.* Braunschweig, 1930.

be arranged in circular order:

(a) $\alpha' = \beta$, $\beta' = \alpha$; (b) $1' = 2$, $2' = 3$, $3' = 1$.

Let us now recursively define a certain function φ in the following manner:

(1) $\varphi(\alpha) = 1$.

(2) If n is an element of the set M, then $\varphi(n') = [\varphi(n)]'$. This definition is contradictory.

According to condition (a) $\alpha' = \beta$. Owing to the symmetry of equation, $\beta = \alpha'$. Then $\varphi(\beta) = \varphi(\alpha') = (\varphi(\alpha))'$ (according to point (2) of the definition). But $\varphi(\alpha) = 1$ (point (1)). So we obtain that $\varphi(\beta) = (\varphi(\alpha))' = 2$ (by condition (b)). But by condition (a) $\beta' = \alpha$. Then by point (2) of the definition we obtain $\varphi(\alpha) = \varphi(\beta') = (\varphi(\beta))'$. But $\varphi(\beta) = 2$. Consequently, $(\varphi(\beta))' = 2'$. But by condition (b) $2' = 3$. Thus $\varphi(\alpha) = 3$. But by point (1) of the definition $\varphi(\alpha) = 1$. Since equality is symmetrical and transitive, we obtain $1 = 3$. By condition, however, the objects of the set N (that is, 1, 2, 3) are different, that is, $1 \neq 3$.

Recursive definitions are applied to series of objects isomorphic with natural number series. Let us cite a certain real analogue of a recursive definition. Assume that our series of material objects is ordered in such a way that the first of them is 3 *cm* long ($\varphi(1) = 3$). Assume that it is known that every object following the n-th object is longer than it by 2 *cm* ($\varphi(y') = y + 2$).

These conditions determine the length of any object of our series.

Let us calculate the length of the fourth object in the series. We can act in a manner prescribed by recursive definitions.

(1) Object 4 is longer than object 3 by 2 *cm*.
(2) Object 3 is longer than object 2 by 2 *cm*.
(3) Object 2 is longer than object 1 by 2 *cm*.
(4) Object 1 equals 3 *cm*.

Now let us go from (4) back to (1) and calculate the length of the fourth object: 3 *cm* + 2 *cm* + 2 *cm* + 2 *cm* = 9 *cm*.

Inductive definitions[1] (in their constructive interpreta-

[1] Inductive and recursive definitions are discussed in Yu. Gastev's contribution 'Recursive and Inductive Definitions' (part of the article 'Definition') and the article by Yu. Gastev, I. Shmain 'Recursive Functions and Predicates' in the *Filosofskaya entsiklopediya* (Philosoph-

tion) and recursive definitions are linked with the problem of the so-called *enumerable and decidable sets. A set is called enumerable* if we can construct any of its elements according to a definite algorithm proceeding from primitive objects and operations. *A set is called decidable* if there is an algorithm permitting to define effectively for any object of the domain whether it belongs to the set as its element or not. The predicate P defining a certain set is decidable if both it and its complement are enumerable.

The same concepts may be defined regardless of the concept of algorithm, by proceeding from the concept of computable (recursive) functions. Decidable sets may be illustrated by the set of the formulas of a calculus introduced by an inductive definition: we have an effective method (an algorithm) to establish, for any finite sequence of the signs of an alphabet, whether we deal with a formula or not.

An enumerable but undecidable set may be illustrated by the set of provable formulas (theorems) of some calculi. If the set of provable formulas (as in the propositional calculus) turns out to be decidable, the problem of decidability for this calculus is said to be solved.

¡Every decidable set is enumerable, since any of its elements may be constructed according to a definite algorithm by applying certain operations to the primitive objects. However, not every enumerable set is decidable. Such is, for example, the set of provable formulas in predicate calculus. It is enumerable since we can construct it by proceeding from the axioms (which are in their turn regarded as provable formulas) and the elementary derivation rules (these may be enumerable by Gödelian numbering). But such a set is undecidable: there is no general effective procedure for establishing the membership of the formula in the set of provable formulas (e.g., through analysing some of its forms) since the set of nonderivable formulas in this calculus is not recursively enumerable.

One of the problems discussed by logicians is that of the *reducibility of implicit inductive and recursive definitions to explicit* 'normal' definitions having the form of equations and equivalences. Here the term, function, or predicate defined do not appear in the definiens.

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ical Encyclopedia). Vol. 4, Sovietskaya Entsiklopediya Publishers, Moscow, 1967.

Inductive definitions may be reduced to normal explicit definitions, e.g., on a set-theoretical basis. This mode of reduction is presented by J. Slupecki and L. Borkowski.[1] Such reduction is based on the concepts of a closed set and the smallest set.

The set Z is called *closed* with regard to a certain operation O if, and only if, the application of the operation O to the elements of this set again generates an element of this set.

The set Z is called the *smallest* of all the sets having the property S, if and only if this set Z has the property S and is also contained in any of the sets having the property S. The inductive definition of formula in the propositional calculus given above (see p. 121) may then be formulated as follows: 'The set of propositional calculus formulas is the smallest set containing the propositional variables p, q, r, p_1, q_1, r_1 . . . which is closed with regard to the operations of negation, conjunction, disjunction, and implication.'

The inductive definition of provable formula given above would then be formulated as follows: 'The set of provable formulas of a given axiomatic system is the smallest set consisting of the axioms of the system and closed with regard to derivation rules accepted in the system.'

The definitions cited here might be written in symbolic form, if we introduced corresponding quantifiers connecting the variables *whose values are sets and not individuals.*

As distinct from the corresponding inductive definitions of formula and provable formula, the explicit definitions cited above are extremely *unconstructive.* It is difficult to use them, e.g., in finding the smallest set among other sets whose number may prove to be infinite. That is why one resorts to inductive definitions in the relevant systems. Those of them that may be transformed into recursive ones contain, apart from effective devices for generating the objects defined (cf. the definition of provable formula), also effective devices for determining whether the given object belongs to the class of definable objects or not (cf. the definition of formula in propositional calculus).

Reducibility of recursive definitions to explicit ones may also be achieved in a different way. This reducibility is

[1] See J. Slupecki, L. Borkowski. *Op. cit.*, p. 147.

essentially based on set theory or predicate calculus of a higher order than the first.

Slupecki and Borkowski cite an example of this sort of reduction of the definition of multiplication to a normal explicit definition.[1]

A recursive definition of multiplication is given:

$$\begin{cases} a \cdot 1 = a \\ a \cdot (n+1) = a \cdot n + a. \end{cases} \tag{1}$$

In the first equation the defined term 'multiplication' is included only in the left-hand part; in the second equation it is contained both in the left and the right parts. A normal explicit definition of multiplication may take this form, for example:

$$a \cdot b = c \equiv \exists f \{\forall a\,(f(a,\ 1) = a) \wedge \forall a\,\forall n\,(f(a,\ n+1) = \\ = f(a,\ n) + a) \wedge f(a,\ b) = c\}. \tag{2}$$

This definition may be read as follows: $a \cdot b = c$ if, and only if, there is a function f such that, for any a and 1, it constructs a and for any a and n, $f(a,\ n+1) = f(a,\ n) + a_1$, and it constructs c out of a and b.

The Dfn of explicit normal definition (2) includes the characteristic of the function f which was unambiguously and constructively defined in the recursive definition (1). In definition (2) the definition of the multiplication operation is reduced to a description of the characteristics of a certain function f.

Inductive and recursive definitions may thus be reduced to explicit ones on a set-theoretical basis or on the basis of an extended predicate calculus. However, this reduction is not expedient, since a number of limitations have to be formulated with regard to explicit definitions of this form (to exclude contradictions and 'infinite' processes). Besides, they are extremely unconstructive.

In 1931 Gödel demonstrated that if the consistency of the natural numbers arithmetic containing the operations 'following n' ($'$), addition ($+$), and multiplication ($\cdot$) is proved ($+$ and $\cdot$ being defined recursively), all the other recursive definitions of arithmetic may be reduced to explicit ones by applying the narrow predicate calculus only.[2]

[1] J. Slupecki, L. Borkowski. *Op. cit.*, p. 146.

[2] E. W. Beth. *The Foundations of Mathematics*. North-Holland Publishing Company, Amsterdam, 1959, pp. 129-31.

'Therefore [writes Beth], we have reasons to believe that every effectively computable function f and every decidable predicate P can be explicitly defined in terms of $+1$, $+$, and $\cdot$'.[1]

3.5. THE PROBLEM OF ELIMINATION OF SYMBOLS AND SIGN EXPRESSIONS INTRODUCED BY DEFINITIONS

We often extend formalised logico-mathematical theories by introducing new symbols and designations through various kinds of the abbreviative explicit definitions, thus obtaining a definitional extension S_2 from the formal system S_1. It has been pointed out that abbreviative definitions may be regarded both as rules and as theorems of the system.

The introduction of new definitions into the system is sometimes linked with the introduction of *new formation rules* and *new defining 'axioms'*. Thus, if expressions $\sim$, $\exists!x$ are introduced into the system S_1 by abbreviative definitions, each time we would actually have to formulate a corresponding formation rule and defining axioms or axiom schemata.

Thus, for the sign $\sim$ we would have to formulate the following formation rule: 'If A is a formula and B is a formula, then $A \sim B$ is a formula.' For the expression $\exists!x$ we would have to formulate the rule: 'If x is a variable and $A(x)$ is a formula, then $\exists!xA(x)$ is also a formula.' For these expressions one would have to formulate the following defining axiom schemata or axioms:

$$\begin{aligned}&[((A \sim B) \rightarrow (A \rightarrow B) \wedge (B \rightarrow A) \wedge ((A \rightarrow B) \wedge \\ &\wedge (B \rightarrow A) \rightarrow (A \sim B))] \\ &\exists! xA(x) \sim \exists x[A(x) \wedge \forall y(A(y) \rightarrow x = y)].\end{aligned} \tag{3}$$

New formation rules introduce new sign expressions into a system, while axioms and axiom schemata introducing them make for their deductive use in the system and at the same time may be regarded as theorems. The class of provable formulas in S_1 would be narrower than the class

[1] *Ibid.*, p. 131.

of provable formulas in S_2, if each formula containing new signs and provable in S_2 could not be transformed into a logically equivalent formula of the system S_1 which is also provable in it. If this conversion is possible, then *definitional extension* of S_1 into S_2 by means of new formation rules and new axioms for introduction *proves to be inessential*: it does not result in such a class of provable formulas in S_2 which would not be provable in the above conversion into S_1. Every new provable formula in S_2 turns out to be logically equivalent to the corresponding formula in S_1. New sign expressions introduced in S_2 thus prove to be eliminable and reducible to combinations of signs appearing in S_2. The defining axioms formulate in each particular case the conditions for *introducing and eliminating* new sign expressions by abbreviative definitions.

Let us designate the derivability relation in S_1 by $\vdash_1$ and the relation of derivability in S_2, by $\vdash_2$. The general conditions for eliminability of new sign expressions from S_2 in S_1 may then be formulated as follows:

'...The new notations or symbols (with their postulates) [1] are eliminable (from S_2 in S_1) if there is an effective process by which, given any formula Σ of S_2, a formula Σ' of S_1 can be found, such that:

(I) If Σ is a formula of S_1, then Σ' is Σ.

(II) $\vdash_2 \Sigma \sim \Sigma'$.

(III) If $\Gamma \vdash_2 \Sigma$, then $\Gamma' \vdash_1 \Sigma'$.'[2]

The formulas Σ' from S_1 constitute a regular subset of Σ formulas from S_2. If these conditions are fulfilled, a number of theorems may be proved about eliminability of new symbols and designations.

Theorem 1: $\Gamma \vdash_2 \Sigma$ then and only then when $\Gamma' \vdash_1 \Sigma'$ (this is an inverse theorem with regard to condition (III), see p. 136).

Proof. If formula Σ' including no new symbols or designations is derivable from Γ' of system S_1 by logical means, Σ' is all the more derivable from Γ' by the means of system S_2 (which includes all the logical apparatus of system S_1),

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[1] That is, axioms or axiom schemata introducing new sign expressions.

[2] S. C. Kleene. *Op. cit.*, p. 405.

that is, $\Gamma' \vdash_2 \Sigma'$. But if Σ' is derivable from Γ' by the means of S_2, then Σ' is also derivable by the means of S_2 from Γ, since Γ in S_2 may also include all the premises from Γ' appearing in S_1 and the corresponding premises including new symbols and designations (that is, if $\Gamma' \vdash_2 \Sigma'$, then $\Gamma \vdash_2 \Sigma'$). But, since the general conditions on eliminability stipulate $\vdash_2 \Sigma \sim \Sigma'$ (see point II), the expression $\Gamma \vdash_2 \Sigma$ will obtain.

Theorem 2. If Γ, Σ are formulas from S_1, then $\Gamma \vdash_2 \Sigma$ then and only then when $\Gamma \vdash_1 \Sigma$.

Proof. If the formulas Γ, Σ are from S_1, then, in accordance with point (I) of the general eliminability conditions, formulas Γ, Σ are formulas Γ', Σ'. The condition of theorem 2 is consequently of the form $\Gamma' \vdash_1 \Sigma'$. But, since this condition obtains, it is proved by theorem 1 that $\Gamma \vdash_2 \Sigma$. The proof of an inverse of theorem 2 is trivial.

In the book cited here Kleene considers the elimination of descriptive definitions and proves corresponding theorems.[1]

Let us now consider the difficulties involved in the elimination of nominal abbreviative definitions in formalised logical languages and formal systems.

Let us consider a few examples.

Example I.

Let system S_1 be given.

The morphology of system S_1 includes the alphabet of the system and the rules for the formation of those expressions which are formulas.

The alphabet: p, q, r, p_1, q_1, r_1, p_2, . . . , $\supset$, 0, (,).

The rules for the formation of formulas:

1) If A is one of the letters p, q, r, p_1, q_1, r_1, . . ., then A is a formula.

2) 0 is a formula.

3) If A and B are formulas, then $(A \supset B)$ is a formula (there is an agreement concerning omission of brackets; e.g., $A \supset B \supset C \supset D$ is interpreted as $(((A \supset B) \supset C) \supset D)$).

The system has usual axioms and permissible deduction rules.

Assuming that the alphabet of the system has been appropriately extended beforehand, as well as the concept of

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[1] *Ibid.*, pp. 407-09.

the formula, let us now formulate the 'defining axioms'

$$\nabla A \equiv (A \supset 0), \tag{1}$$
$$\vee (A, B) \equiv ((A \supset B) \supset B), \tag{2}$$
$$\wedge (A, B) \equiv \nabla (A \supset \nabla B) \tag{3}$$

(the sign $\equiv$ is read 'is by definition'; why we do not directly use the sign $\sim$ will be discussed somewhat later). We have thus obtained a definitional extension of S_1. Let us designate it S_2.

Substitution of notation, e.g., of $\vee(A, B)$ for $(A \vee B)$, is performed by agreement on another form of notation including no new signs as compared to the original one, rather than by definition.

Assume that we have in S_2 the expression

$$((p \vee q) \supset r) \wedge (\nabla (p \vee q) \supset r). \tag{4}$$

Let us designate it by X:

$$X \overline{\circ} ((p \vee q) \supset r) \wedge (\nabla (p \vee q) \supset r), \tag{5}$$

where X is a metasign and $\overline{\circ}$ an auxiliary sign for denoting expression (4).

Let us designate $((p \vee q) \supset r)$ by Y_1 and $(\nabla (p \vee q) \supset r)$ by Y_2.

$$X \overline{\circ} \wedge (Y_1, Y_2). \tag{6}$$

The question now arises whether we can directly reduce expression (4) to combinations of atomic objects of the system S_1 by applying our axiom schemata (1)-(3)?

Expression $\wedge (Y_1, Y_2)$ cannot be directly reduced to combinations of the signs of the system S_1, as there is no corresponding axiom schema for this purpose. Schema (3) does not apply here, as its Dfn also contains the sign ∇ which we find only in S_2. In Curry's words, such objects as $\wedge(Y_1, Y_2)$ are not 'simple objects' of the system S_2.

Y_1 cannot be directly eliminated either, by reducing it to combinations of atomic objects of the system S_1, as there is no special defining axiom schema permitting to effect the required reduction directly.

Let us write Y_1 in this way: $Y_1 \overline{\circ} \supset (\vee (p, q), r)$, designating then the expression $\vee (p, q)$ in brackets by Y_{11}.

Y_{11} is no longer a 'simple object' of system S_2 as its elimination requires axiom schema (2).

By applying axiom schema (2), we may represent Y_{11} in the form of the expression $(p \supset q \supset q)$.

Then $Y_1 \overline{\circ} p \supset q \supset q \supset r$.

We have thus reduced Y_1 to the objects and operations of the system S_1.

Let us now reduce Y_2 to the objects and operations of the system S_1.

$Y_2 \overline{\circ} \supset (\overline{\ } (p \vee q),\ r)$.

Let us now designate $\overline{\ }(p \vee q)$ by Y_{12}. Then $Y_2 \overline{\circ} \supset (Y_{12}, r)$.

The object Y_{12} is not a simple object.

Therefore let us first eliminate $(p \vee q)$. According to axiom schema (2) we shall obtain $(p \supset q \supset q)$. Y_{12} as a whole has the form $\overline{\ } (p \supset q \supset q)$. Applying axiom schema (1) to the latter expression, we shall obtain:

$p \supset q \supset q \supset 0$.

Thus

$Y_{12} \overline{\circ} p \supset q \supset q \supset 0$,

$Y_2 \overline{\circ} \supset ((p \supset q \supset q \supset 0),\ r)$,

$Y_2 \overline{\circ} p \supset q \supset q \supset 0 \supset r$.

Then expression (6) will assume the form:

$X \overline{\circ} (Y_1,\ Y_2) \overline{\circ} \wedge [(p \supset q \supset q \supset r),\ \ (p \supset q \supset q \supset 0 \supset r)]$.

But, to eliminate the sign $\wedge$, one cannot apply axiom schema (3) directly to this expression. To achieve this, one first has to form a negation of Y_2, then add Y_1 to the left of the negation of Y_2 linking it with the latter by the implication sign, and only after that put the negation sign before the entire expression. We shall thus obtain:

$\overline{\ } (Y_1 \supset \overline{\ } Y_2)$

To eliminate the negation sign, one may apply axiom schema (1) to the expression $\overline{\ } Y_2$. The same operation may then be performed on the expression $\overline{\ }(Y_1 \supset \overline{\ } Y_2)$.

As a result, we shall obtain the formula

$[(p \supset q \supset q \supset r) \supset (p \supset q \supset q \supset 0 \supset r \supset 0)] \supset 0$.

Let us consider example II.

Assume that the system is extended by the defining axiom $F(f) \equiv \overline{\ } f(f)$ (cf. the definition of the set of sets that do not

contain themselves as their own element, see 2.3). The variable f may here range over the sets and the corresponding predicates. If we then substitute F for the variable f, we shall obtain $F(F) \equiv \neg F(F)$ (a). To eliminate the Dfd, that is, the expression $F(F)$, which occurs both in the left-hand and the right-hand parts of (a), let us replace it by the entity which it equals by definition (i.e., $\neg F(F)$).

We shall obtain successively:

$$\neg F(F) \equiv \neg\neg F(F),$$
$$\neg\neg F(F) \equiv \neg\neg\neg F(F),$$
$$\neg\neg\neg F(F) \equiv \neg\neg\neg\neg F(F),$$

etc., ad infinitum.[1]

The expression introduced by the defining axiom cannot be eliminated in this case.

Example III.

Assume that in the above example I of definitional extension of the system S_1 we had only two defining axiom schemata, namely, (2) and (3). We would then be unable to eliminate from the expression $((p \vee q) \supset r) \wedge (\neg(p \vee q) \supset r)$ the signs introduced by definitional extension and reduction to a combination of objects and operations of the system S_1.

The examples considered here demonstrate that there are three cases possible in definitional extension of the original system:

1) new symbols and designations introduced by abbreviative definitions (and corresponding axiom schemata or axioms) may be eliminated or reduced to combinations of symbols and designations of the original system (that is, to the terminal definiens);

(2) new symbols and designations introduced by abbreviative definitions generate infinite processes when an attempt is made to reduce them to terminal definiens;

(3) combinations of symbols and designations introduced into a system obtained by definitional extension cannot be reduced to terminal definiens, that is, to the combinations of symbols and designations of the original system (the reduction process is blocked).[2]

[1] See H. Curry, R. Feys, W. Craig. *Combinatory Logic*. Vol. 1. Amsterdam, 1959.

[2] See H. Curry. *Foundations of Mathematical Logic*. McGraw-Hill, N. Y., 1963, p. 186.

A definitional extension is always *unessential* and the objects of a definitionally extended system are reduced to a *terminal definiens* if the new symbols and designations are introduced by *successive* explicit *noncontradictory* definitions. Contradictory definitions of the $F(f) = \neg f(f)$ type should be excluded. The first definition D_1 extending the system introduces a new sign expression as the Dfd, and the Dfn is the terminal definiens. If the original system was the system S_1, then the system extended in this way will be the system S_2. The definition D_2 extending the system S_2 to S_3 introduces a new sign expression as an abbreviation for the combination of symbols S_2, etc.

The new symbols and designations introduced by definitions are included in the language of the system (extend it), if they are correlated with certain logico-mathematical meaningful concepts and objects in the system's interpretations. As they exist independently of and prior to such systems, the interpreted definitions introducing them will be *analytic*. As these definitions introduce new symbols that extend previously existing systems (before their interpretation is given), they must be regarded as *synthetic*. If a new expression is introduced into the language by the symbol $\overline{\bigcirc}$ for temporary convenience, this expression is not regarded as extending the given system. The sign $\underline{\overline{\bigcirc}}$ (the sign of graphical equality) does not introduce any new objects in the system. In example I considered above the signs X, Y_1, Y_2, Y_{11}, Y_{12} played precisely this role of signs introduced for temporary convenience.

The signs and sign expressions $\sim$, $\exists!x$ considered above are means of extending the corresponding systems: not only axioms and axiom schemata are formulated for them, but the alphabet of the system and the inductive definition of the formula are extended. Curry calls the extension of the original system a *proper* extension if for each new Dfd there is at most one defining axiom corresponding to an explicit noncircular definition. Difficulties in definitional reduction arise when defining axioms (as in example I) are used to introduce a system of signs and designations, the same new signs appearing in the Dfd of one axiom and in the Dfn of another or both in the Dfd and the Dfn of the same defining axiom (e.g., in recursive definitions). The new symbols introduced into the system may then prove to be irreducable to the terminal definienses.

Kleene gives an example of the functional symbol '·' (multiplication), together with the recursive equations defining it, being uneliminable from a certain formalism S_2 formed by this symbol's addition to the system S_1. The system S_1 is simply consistent and its decidability solvable. If '·' were eliminable and reducible to the terminal definiens, the system S_2 would also be simply consistent and the problem of decidability solvable. However, that is not the case.[1]

The analysis of eliminability of symbols and sign expressions introduced by explicit nominal definitions *makes it possible to distinguish between the meaning of the sign* $\equiv$ ('is by definition', 'equals by definition') and $=$ *(the equality sign)*, in accordance with logical equivalence, where propositions or propositional functions are written to the left and to the right of the sign $\equiv$.

Introducing a new symbol by $\equiv$ ('is by definition') instead of some other combination of symbols, we only introduce, generally speaking, an agreement on the substitution of the Dfd for the Dfn in any contexts. However, *since in definitional reduction it so happens that both the Dfd and the Dfn (as well as contexts differing from each other by the presence of either the Dfd or the Dfn) are reduced to an identical terminal definiens, are equated with it*, we can replace the sign $\equiv$ by the sign $=$ (resp. by the sign of logical equivalence).

Considering definitions in deductive systems, Kotarbinski discussed the so-called *creative* and *noncreative* nature of abbreviative explicit definitions (he calls them 'substituting definitions'). A definition extending a system will be creative when used to prove propositions in extended system that cannot be proved without it in the original system. He believes that most scholars nowadays are inclined to reject the idea of substituting definitions playing a creative role: they may be viewed as abbreviations or equivalent expressions that can always be dispensed with, in principle.[2]

There is no doubt that definitions are noncreative when the new symbols and expressions of the system S_2 extended by definitions are eliminable and reducible to the symbo-

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[1] See S. C. Kleene. *Op. cit.*, p. 407.
[2] Tadeusz Kotarbinski. *Wybor pism*. Warszawa, 1958.

lism of the system S_1. This reduction, however, is not always realisable.

Nondeductive experimental nonformalised sciences are usually constructed in such a way that definitions introduced in the development or presentation of a scientific discipline are *creative*, we may use them to prove and substantiate that which cannot be proved without them. In these definitions, objects included in the Dfd and the Dfn often equally extend the stock of the primitive terms of the theory.

3.6. THE PROBLEM OF DEFINABILITY OF TERMS IN FORMALISED THEORIES AND FORMAL SYSTEMS

There is analogy between two groups of concepts in deductive formalised theories and formal systems:

'axiom'—'undefined (primitive) concept',
'theorem'—'definable concept',
'rule of inference'—'rule of definition',
'proof'—'definition'.[1]

Specificially this analogy also prompts similarity between certain problems arising with regard to these two groups. Of this nature is, for instance, the problem of completeness and independence of axioms, on the one hand, and the problem of definability and mutual independence of the primitive terms, on the other.

It is known, for instance, that in a certain system of axioms S_1 the axiom A_k may be superfluous, that is, it may be proved on the basis of the remaining axioms A_1, A_2, . . ., A_{k-1}, A_{k+1} (S_0). Such proofs may be given for any axiom of S_1 on the basis of the remaining axioms of the system S_0. These proofs are called the proofs of independence of the system of axioms S_1 with regard to the system of axioms S_0.

In a similar way, it may so happen that a certain primitive term t_k may be definable in the system S_1 through the remaining primitive terms t_1, t_2 . . ., t_{k-1}, t_{k+1} of the system S_0. It is stated in such cases that the term t_k of the system

[1] See A. Tarski. *Logic, Semantics, Metamathematics*. Oxford at the Clarendon Press, 1956, p. 296.

S_1 is definable through the terms of the system S_0. In other words, t_k may be defined in the system S_0 through t_1, t_2, . . ., t_{k-1}, t_{k+1}. It is taken into account that everything that could be deduced in the system S_0 will also be deducible in S_1. If the primitive terms of the system S_1 are undefinable through the remaining primitive terms of the system S_0, the terms of the system S_1 are said to be mutually independent. The proofs for definability (resp. mutual independence) of the theory's primitive terms were suggested in 1899 by A. Padoa (the method of proving the independence of axioms being first applied in 1868 by E. Beltrami).[1]

Over the last decades the problems of definability have been dealt with by a number of authors. The basic results were obtained by E. Beth and W. Craig. Beth gave the formulation and the first substantiations of the theorem concerning the so-called *provable definability*.[2] This theorem is formulated with regard to n-place fixed predicates (predicate parameters) α, t_1, t_2, . . ., appearing as primitive terms in the restricted predicate calculus and some of its extensions. This theory is formulated on the basis of two fundamental concepts, that of *the explicit definability of the primitive term* and that of *the implicit definability of the primitive term*.

Assume that our theory has the axioms φ that are closed formulas containing predicates or predicate parameters α, t_1, t_2, t_3 . . . . We shall say that α is *explicitly definable* with regard to φ in terms of t_1, t_2, t_3, . . ., if and only if we can find the formula $U(x_1, x_2, \ldots, x_n)$ containing only the terms t_1, t_2, t_3, . . ., such that the following semantic equivalence will obtain:

$$\forall x_1 \forall x_2, \ \ldots, \ \forall x_n \, (\alpha(x_1, x_2, \ldots, x_n) \sim U(x_1, x_2, \ldots, x_n)).$$

The left-hand and the right-hand sides of the equivalence are equally interpretable: they are satisfied by precisely the same models.

It will be said that α is *implicitly definable* if and only if ψ is a set of formulas obtained from the set of formulas φ by replacing every instance of α by an n-place predicate b (which does not appear in φ) such that the union of φ

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[1] See E. W. Beth. *Op. cit.*, pp. 87-89.
[2] *Ibid.*, pp. 288-93.

and ψ (that is, '$\varphi \cup \psi$') yield the formula

$$\forall x_1 \forall x_2, \ldots, \forall x_n [\alpha(x_1, x_2, \ldots, x_n) \sim b(x_1, x_2, \ldots, x_n)].$$

The theorem of provable definability itself is formulated as follows: the predicate parameter α is explicitly definable with regard to φ in terms of $t_1, t_2, \ldots$, then and only then when it is *implicitly* definable with regard to φ through $t_1, t_2, \ldots$.

Assume that the primitive terms are α, t_1 and t_2. The system of axioms φ consists of one single axiom

$$t_1(x) \supset (\alpha(x) \sim t_2(x)).$$

Let us designate this axiom by A. The question arises whether α is an implicitly definable predicate with regard to φ by t_1 and t_2. To answer this question, let us substitute b for α in our axiom A. We shall obtain a new system of axioms φ'. Axiom A will be transformed into $t_1(x) \supset \supset (b(x) \sim t_2(x))$. Let us designate the axiom from φ' by A'.

The predicate α will be implicitly definable if and only if the following expression obtains:

$$\forall x [((t_1(x) \supset (\alpha(x) \sim t_2(x))) \wedge (t_1(x) \supset (b(x) \sim \sim t_2(x)))] \vdash \forall x (\alpha(x) \sim b(x))$$

or if (which is much the same) the following formula is derivable from the conjunction $A \wedge A'$:

$$\forall x (\alpha(x) \sim b(x)).$$

But this formula is not derivable from $A \wedge A'$. Therefore, the predicate α is not *implicitly definable* with regard to the system of axioms φ in terms of t_1 and t_2. Consequently (according to our theorem) the predicate α is not explicitly definable with regard to φ in terms of t_1 and t_2.

Beth's theorem about provable definability is proved on the basis of the so-called Craig theorem (it is sometimes called 'interpolation theorem').

This theorem may be formulated as follows: for any derivable formula $A \rightarrow C$ of the classical logic of one-place predicates there exists the formula B which includes only such free individual and predicate signs as appear both in A and in C, the formulas $A \rightarrow B$ and $B \rightarrow C$ being derivable from B.[1]

[1] For proof of this theorem see R. C. Lyndon. *Notes on Logic*. Van Nostrand, Princeton (N. J.), 1966, pp. 112-14; K. Schütte. 'Interpolation Theory for Intuitionist Logic of Predicates'. *Matemati-*

It follows from Beth's theorem that Padoa's method is applicable in establishing the independence of the predicate α.

In Aristotelian syllogistics, as we know, the connection between the extreme terms in the consequent is realised through a certain set of relations of the middle term to the extreme terms in the antecedents. It appears that some consequences from the Craig theorem may be regarded as a generalisation of the various cases of the relation of the middle term in the antecedents to the extreme terms, a generalisation substantiating the inference of the consequent $A \rightarrow C$ from the premises $A \rightarrow B$ and $B \rightarrow C$.

A similar result was obtained for the extended predicate calculus by Tarski in 1934. Taking into account the above theorem on provable definability, Beth presents this result in the following way.[1] Assume that we have already proved the formula (c). Let $A(X)$ be the formula obtained by the replacement of each occurrence of α in the formula A by the corresponding predicate variable X. Then the formula

$$X[A(X) \rightarrow X(x_1, x_2, \ldots, x_n)] \sim \alpha(x_1, x_2, \ldots, x_n)$$

may be viewed as an explicit definition of α. This kind of definition may be illustrated by the explicit definition of multiplication obtained above by transforming the corresponding implicit recursive definition of multiplication (see p. 134).

The definability problems considered here are usually included in the problem of the so-called syntactical definability. Here also belongs the definability of terms newly introduced into a theory by explicit definitions. The new term is in such cases regarded as definable if it may be represented (in any context of the given theory) as a combination of the primitive symbols of the theory (that is, if the definitional reduction may be brought to the stage of 'terminal definienses', see 3.5). If explicit definitions are introduced into a theory in a certain standard way (see 3.5) and the newly introduced term is definable through the primitive terms then the proposition which is an explicit

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cheskaya teoriya logicheskogo vyvoda (The Mathematical Theory of Logical Inference), Moscow, 1967, pp. 285-95; V. A. Smirnov. *Formalny vyvod i logicheskiye ischisleniya* (Formal Inference and Logical Calculi), Nauka Publishers, Moscow, 1972, pp. 151-55.

[1] See E. W. Beth, *Op. cit.*, p. 292.

definition including the sign $\equiv$ is a theorem of the system, where the sign $\equiv$ is replaced by the sign of logical equivalence or equality. A different concept of definability is also analysed in modern logic, one that is called semantic definability (Gödel), or λ-definability (Church). A discussion of this extensive range of problems goes beyond the scope of the present work. We shall restrict ourselves to some brief observations on this score.

Semantic definability of a certain object (including abstract objects of higher levels of abstraction than the first) studied by a formalised theory is interpreted as its unambiguous description, its representation by the means of a given formalised theory. In other words, an object studied by a formalised theory is semantically definable if and only if a corresponding sign model may be constructed for it in the given theory. Thus it may be said that a k-place relation R is semantically definable (expressible) in the given theory T, if its means may be used to describe a certain k-place predicate $P(x_1, \ldots, x_k)$ (where $x_1, x_2, \ldots, x_k$ are free variables) which certain terms $a_1, \ldots, a_k$ satisfy then and only then when the relation R obtains between $a_1, \ldots, a_k$ taken in the same order.

Cases similar to semantic definability appear already in the initial stages of studying logic (for instance, when we prove the theorem of functional completeness of propositional logic). Its essence is precisely this: any Boolean function is expressible in terms of the language of this theory.

An important result in this field was obtained by Tarski. He proved a theorem to the effect that for a very extensive class of formalised theories (at any rate for those which satisfy the conditions of Gödel's incompleteness theorem) the truth predicate of a formula of the given theory T is inexpressible in the theory T.[1]

[1] See R. C. Lyndon. *Op. cit.*, pp. 123-124; A. Fraenkel, Y. Bar-Hillel. *Foundations of Set Theory*. Amsterdam, 1958, Chapter V, § 6.

CHAPTER 4

DEFINITIONS IN PHYSICAL THEORIES

4.1. THE SPECIFICITY OF DEFINITIONS IN PHYSICS

Chapters 1 and 2 considered problems concerning the general theory of definitions. Illustrations for analysis were in many cases drawn from definitions traditionally used in mathematics and logic (e.g., implicit, contextual, descriptive, nonpredicative and other kinds of definition). The specificity of definitions in mathematics was considered as well as of the kinds of definition that are specific for mathematical activity.

Definitions in mathematical disciplines are constructed in such a way as to make them effective means of solving mathematical problems (e.g., those involving proofs of theorems, calculation of functions, postulating enumerable decidable sets, etc.). Definitions in mathematics are an effective means of constructing mathematical theories and developing them deductively. Definitions (e.g., those with ι-operators and ε-operators) introduce into mathematical theories abstract and idealised objects extending the range of mathematicians' activities and making them more effective (e.g., imaginary numbers are introduced in this way).

Definitions in mathematics are thus a powerful means of solving its problems and developing its theories, not merely a means of introducing, characterising, specifying and clarifying something. The situation in physics (and other fields of mathematical science) is much the same.

When the natural sciences were still at the descriptive level and quantitative analysis did not yet play a dominant role, the principal problem was that of establishing sufficiently reliable criteria for identifying and distinguishing the objects under consideration by their essential properties (resp. for distinguishing between and identifying the meanings of the terms). Definitions played an important

role in solving this problem, making it possible to classify the objects under analysis and to establish certain qualitative dependences between them. We encounter such classificatory definitions in defferent spheres of our life, for example, in mastering our native language. Classificatory definitions taking the form of definitions *per genus proximum et differentiam specificam* were described already, as we know, by Aristotle. To these were later added qualitative genetic definitions.

Physics began to develop rapidly from the time when it started to register experimental results on the basis of measuring the corresponding physical quantities (mass, length, power, speed, acceleration, etc.) that are abstract objects; when it began to formulate the laws of nature on the basis of certain general idealisations presupposing the introduction of idealised objects into science; when it began to apply such means of recording discovered correlations and laws which enabled it in many cases to obtain the necessary data by experimental measuring procedures, by computation. In a situation of this sort it is naturally not enough to define the objects under study by merely indicating their properties (even if they are essential) that would permit their specification. Definitions must here be formulated in such a way as to create the conditions for fruitful activity at the various levels and stages of physical science. The physical objects introduced by definitions should first of all contain a method for introducing them and for singling them out on the basis of experimental measurement procedures, as well as a method for eliminating them on the basis of applying theory in practice. They should serve as the means of connecting the propositions formulated in the language of mathematics with the physical reality, the latter being given to the physicist as a collection of objects that are observed and measured in an experimental situation (the physicist disengages himself from the other aspects of the objects included in the experiment). Definitions in physics, just as in mathematics, are therefore formulated with the view to ensuring the effective solution of its problems.

It is a known fact, however, that the structure of physical sciences is more complex than of the mathematical disciplines. This can be seen already in the fact that we deal with several levels of physical activity: experimental and

measuring (including that of formulating empirical laws), theoretical and interpretative. At different levels of activity we have to use definitions to introduce different objects. The general criterion of advisability of using definitions to introduce particular objects is practice, which here appears as experimental measuring procedures (we have in mind those cases, too, when the consequences deduced in a theory including new objects are confirmed by experiment and observation). Of considerable importance is also the criterion of agreement between the theories which include new objects and previous theories that do not contain them but have been tested by practice in the broad sense.

To illustrate our point, let us consider the following example.[1] The definition of mass that is still shared by many physicists was formulated already by Newton in his book *The Mathematical Principles of Natural Philosophy*: '*The quantity of matter is the measure of the same, arising from its density and bulk conjunctly*'[2] (i.e., $m=p\cdot v$, where p is density and v, volume). This definition, however, cannot be regarded as a correct operational definition without further qualifications: it is circular since density according to Newton, is defined through mass ($p=\frac{m}{v}$) and there is no definition here of mass independent of density and volume of the body.

Still, Newton insists on defining mass as the quantity of matter, pointing out that it may be defined by the weight of the body to which it is proportionate (thereby eliminating the vicious circle in the definition cited above), as in his view the idea of inertia may be substantiated in this way.

Newton defines force as follows: '*This vis infita, or inner force of matter, is a power of resisting, by which every body, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line.*'[3] Newton adds that 'this force is ever proportional to the body whose force it is; and differs nothing from the inactivity of the mass, but in our manner of con-

[1] See L. B. Bazhenov. 'The Problem of Matter in Physics and Philosophy'. In: *Filosofiya yestestvoznaniya* (Philosophy of Natural Science), Politizdat Publishers, Moscow, 1966, pp. 31-41.

[2] I. Newton. *The Mathematical Principles of Natural Philosophy*. Vol. I, Knight and Compton, London, 1803, p. 1. (Italics mine—*D.G.*)

[3] *Ibid.*, p. 2. (Italics mine—*D.G.*)

ceiving it.'[1] Here Newton actually proceeds from a concept of mass as the measure of inertia. Since inertia is the inner force of matter, the greater the quantity of matter (viz., the greater the mass), the greater the force of resistance. Proportionality of the measure of inertia to the weight as well is perceived as an experimental fact, only to be explained in the general theory of relativity. The definition of mass as the measure of inertia is included in the formulation of Newton's second law: $F=m\cdot a$. The conception of mass as quantity of matter, being devoid of any operational meaning, becomes redundant for physics.

The development of physics itself has shown the meaninglessness of defining mass as quantity of matter. Suffice it to point out that the mass of one and the same electron moving at different velocities with regard to different frames of reference will be different. This cannot be correlated with the definition of mass as quantity of matter.

For several centuries (beginning with Galileo and Newton) physicists have used a special type of definition different from that usually described in books on traditional logic and constituting a specific feature of mathematical natural science. However, problems involved in the use of such definitions began to be investigated in the logic and methodology of science only at the end of the 19th century.

This problem is thoroughly discussed in Percy Bridgman's book *The Logic of Modern Physics* (New York, 1949).

4.2. BRIDGMAN'S CONCEPTION OF OPERATIONAL DEFINITIONS

The concept of operational definitions was introduced into logic and the methodology of natural sciences by Percy Bridgman, the outstanding American experimental physicist and Nobel Prize winner. To explain the operational mode of introducing concepts into physics by means of definitions, he analysed the concept of length. In what way do we determine length? In the simplest cases (e.g., in measuring the length of a house or a plot) we fix the ends A and B of the object to be measured, and then

[1] *Ibid.*

take a suitable measuring instrument (e.g., a tape-line) and stretch the line so as the first marking on it coincides with A and some other marking, with B. Then we read the length AB as indicated by the tape-line.

This operation seems extremely trivial in principle, but it proves to be complicated enough in practice when we try to attain the necessary precision of measurement. Thus we have to be certain that the tape-line is not stretched when it is extended between A and B, that its length does not change with temperature, that it is correctly fitted to the measured object, etc. Considerable mistakes are possible even in this simple case of measuring.

Let us now assume that we have to measure a moving object, say the length or height of a moving carriage. This task is more complicated than the previous one. When new and more complex tasks arise, we attempt to reduce them to more elementary ones, i.e., those that we are already capable of solving. It is natural to reduce the above task to that involved in measuring of the lengths of sufficiently rigid stationary objects. This can be done practically by someone jumping on a moving carriage and taking the measurements and then stopping the carriage and doing the same. The measurement results will be identical.

The process of measuring becomes exceedingly complicated when we deal with measuring the length of objects moving at very high speeds. Here again, however, we attempt to reduce these complex tasks to the simpler ones of measuring the lengths of stationary objects. In Bridgman's words, that was precisely what Einstein primarily tried to do in his mechanics. However, Einstein's methods for measuring the lengths of bodies moving at great speeds are motivated not only by his desire to perform the kind of reduction indicated above but also by considerations of simplicity and naturalness in applying the corresponding mathematical apparatus to measuring length.

This is how Bridgman presents Einstein's measuring procedure. The observer measuring the lengths of bodies travelling at high speeds should imagine a plane in which a body moves (for simplicity sake a two-dimensional space is considered) in the following manner. A clock is placed in each point in the plane in which movement occurs, all the clocks being synchronised. It is assumed that there is an observer near every clock and that those of them who

happen to be at the ends of a moving body (of its length) can register its position at the same time. The length registered in this way may then be measured by the usual elementary methods. This measuring procedure, however, is based on the concept of simultaneity of events (the ends of the length of a moving body, in this case). As the concept of simultaneity of events is relative, depending on the movement of the observer's frame of reference, the body's length also proves to be a function of its velocity. But length in the sense of Einstein and length in the elementary sense are interconnected. Thus, 'in either case in the limit, as the velocity of the measuring system approaches zero, the operations approach those for measuring the length of a stationary object'.[1]

Similar extrapolations may be realised in various directions. Objects of various sizes are measured—both large and small. In measuring large areas of land with a theodolite we use different operations from those applied in measuring the length of a house. Besides, we also resort to a definite set of computation procedures, assuming that the sunrays are absolutely straight and that Euclidean geometry applies. This measuring procedure may be replaced by elementary measurement with the aid of a rigid scale. The results will be the same. That is why we do not feel the need for introducing a new name for optical length if we measure it on Earth. However, when distance between stars is measured optically, the situation is radically changed. The optical methods cannot be replaced by direct ones, and the results of measurements cannot be compared. Space here is of an essentially optical nature. We cannot be certain that it may satisfactorily be described by Euclidean geometry. Therefore, says Bridgman, the propositions 'The distance between such and such stars is five light years' and 'I am 6 feet tall' are based on essentially different concepts of length.

When distances become extremely small, of the order of elementary particles, the situation is also fundamentally different as compared to direct measurement by rigid scales and the situations that are reducible to such measurements. The situation is made more complex by the fact that the

[1] P. Bridgman. *The Logic of Modern Physics*. The MacMillan Company, N.Y., 1949, p. 13.

objects measured are dialectical and extremely indefinite. What is the possible meaning of the statement that the electron's diameter is 10^{-13} *cm*? Bridgman asks. And he answers that this number appeared from solving certain equations obtained from the equations of electrodynamics by substituting certain experimentally obtained data in them. The concept of length is changed in such a way that it includes the theory of electricity implemented in field equations and, which is most important, it is predicated on the correctness of extrapolating these equations from the domains where they have been tested experimentally to those where their correctness is one of the most fundamental and most disputed problems of modern physics. Bridgman writes: 'As a matter of fact, the concept of length disappears as an independent thing, and fuses in a complicated way with other concepts, all of which are themselves altered thereby, with the result that the total number of concepts used in describing nature at this level is reduced in number.'[1] Bridgman further analyses other fundamental concepts of physics (time, mass, force, etc.) in the light of the operational approach.

What does he mean by operational definitions? *These are definitions of physical magnitudes through descriptions of the set of experimental measuring operations specifying them.* He identifies the concept of physical magnitude with a set of experimental measuring operations. 'In general,' he writes, 'we mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations. If the concept is physical, as of length, the operations are actual physical operations, namely, those by which length is measured; or if the concept is mental, as of mathematical continuity, the operations are mental operations, namely, those by which we determine whether a given aggregate of magnitudes is continuous.'[2] Since one and the same physical magnitude is measured by different methods, their description specifying a corresponding physical magnitude, it follows that in all these cases we shall be dealing with *different concepts of a physical magnitude.*

This view seems to us unnatural, a hindrance to the de-

[1] P. Bridgman. *Op. cit.*, p. 22.
[2] *Ibid.*, p. 5.

velopment of science, as it prevents us from formulating laws of maximal generality: one would have to continually introduce into the formulas various parameters indicating the mode of measuring the magnitudes appearing in them. In that case we would be dealing with a multitude of laws instead of one law. If the operational viewpoint were to be extrapolated to all mental concepts disregarding the relevant identification rules, mathematical apparatus would lose its generality: the values of magnitudes obtained by applying certain formulas (with the aid of one type of paper-and-pencil operations) could not be compared with the values of corresponding magnitudes obtained by applying other formulas (with the aid of other paper-and-pencil operations).

However, Bridgman's extreme 'maximalistically operational' statements (of the type cited above) are not yet sufficient reason for ascribing him such unnatural views. The point is that in applying the concept of operational definition he proceeds from three principles of identification: he does not formulate the first two explicitly, implying them to be self-obvious, and introduces the third through examples.

The first identification principle is the principle of extensionality according to which different descriptions (and the corresponding concepts) are identical (indistinguishable) if their extent is the same, that is, if they are unambiguously correlated with the same objects or sets of objects. Thus, two mathematical descriptions or concepts, 'equilateral rectangle' and 'right-angled rhombus' may be identified as they are extensionally equal: they correspond to the same set of objects called squares. As applied to physics, this principle is formulated by Bridgman in the following way: if two sets of experimental measuring procedures yield the same result with respect to one and the same physical magnitude of one and the same measured object (e.g., measuring length by a rigid scale and theodolite), the descriptions of these procedures and the relevant concepts may be identified.

If we assume that the physicist *knows already what he is measuring* in an object, this principle may be written as follows:

$$C_1(a,\ b,\ m) \wedge C_2(a,\ c,\ n) \wedge b \neq c \wedge m = n \Rightarrow C_1 = C_2. \quad \text{(I)}$$

The concepts C_1 and C_2 are functions of the set of measuring procedures applied (b in one case and c in the other) and of the results of measurement (m in one case and n in the other); but a, the measured physical magnitude of one and the same object in each of the expressions, is a parameter (a fixed variable). Then, if $b \neq c$ but $m = n$, we may identify the concepts C_1 and C_2.

Thus we may insist that we are dealing with the same concept of a certain object's fixed length if we establish its magnitude (1) by a rigid scale, (2) by light waves, (3) by sound waves, (4) by radar devices, obtaining an identical result in all these cases, with $b \neq c$, but $m = n$.

The second principle of identification may be called *the principle of abstraction.* It may be written down like this:

$$C_1(a_k,\ b,\ m) \wedge C_2(d_k,\ c,\ n) \wedge a_k \neq d_k \wedge b = c \wedge m \neq n \Rightarrow C_1 = C_2. \qquad (11)$$

This principle may be formulated as follows: if identical experimental measuring operations b and c are applied in studying the objects a_k and d_k differing in size and the results obtained by the operations are different, the concepts C_1 and C_2 may be identified. The subindex k in a_k and d_k points out that despite a and d being different objects with respect to the characteristic measured (e.g., length), both of them possess this characteristic. In other words, if we measure the length of a section of a road or the length of a house, we may insist that in both cases we deal with the concept of length despite the difference in the result of measurement (the section of a road may be 1,000 *m* long and the house, 30 *m*).

We always use a similar principle of abstraction at the qualitative level of concept formation, too. For instance, in forming the concept of man, we identify all men by the feature of articulate speech. It is common knowledge, however, that individuals share this feature in varying degree. The above principle of abstraction is the principle of operational 'glueing together' of essentially identical but quantitatively different characteristics of physical magnitudes.

The third principle of identification is formulated by Bridgman *for extremely complicated cases of measuring physical magnitudes.* It includes a certain procedure for reducing the more complicated cases to simple ones (e.g., reducing

the procedure of measuring lengths moving at high speeds to measuring elementary lengths).

We shall be dealing with defferent concepts only on condition that *none* of the principles for identifying the different specifying experimental measuring procedures are satisfied. The concept of satisfying is here used in a broader sense than in mathematics. Thus, according to Bridgman, the third principle is not satisfied when the procedure for reducing the more complicated cases to simpler ones is merely *not discovered.*

It follows from the above that Bridgman's interpretation of the concept (even with the earlier reservations about the identification principles) is extremely limited, as the operational approach is here raised to an absolute. Indeed, Bridgman insists, for instance, that the concepts of elementary length and of cosmic length are different concepts, as there are no procedures for reducing the former to the latter. However, such a rigid inference does not follow from Bridgman's conception. What does follow from it is *the conclusion of the possibility* of two different concepts of length, since it is not proved that there is no procedure for reducing cosmic length to elementary length.

It is not quite clear whether Bridgman includes among operational definitions only 'purely' operational definitions, i.e., those describing experimental measuring procedures, or whether he also means definitions describing computational procedures, that is, paper-and-pencil operations. Neither is it clear whether he believes that at some levels of physics definitions are possible other than operational ones. If they are, in what degree and in what sense are they justifiable?

Let us consider these questions.

When Bridgman stresses the operationally empirical nature of science, he is inclined to take an extremely narrow view of definitions, including only descriptions of experimental measuring procedures among them (e.g., in defining force as something measured by a dynamometer, speed as something measured by an accelerometer, etc.). However, in illustrating the operational approach in physics by the above examples, Bridgman often deviates from operational empiricism, including among operational definitions those which rely on computational (paper-and-

pencil) operations as well as experimental measuring procedures. Such is the above definition of length with the aid of a theodolite. Moreover, he also includes among operational definitions descriptions involving computation of the values of physical magnitudes characterising certain objects according to the relevant formulas, as the values of the variables in these formulas are obtained experimentally by applying some measuring operations. This case may be illustrated by the definition of electron diameter.

However, Bridgman is on the whole inclined to regard definitions of physical magnitudes in a generalised and abstract form as mathematical and not physical definitions, although he never formulates his views on this score clearly or definitely.

Operational definitions are widely used in physics and have at least two indisputable advantages. One of these has already been pointed out—their effectiveness and close link with physical activity. The second advantage is that they define more abstract entities through less abstract and more concrete ones that are immediately given and testable. Operational definitions are similar in this respect to contextual definitions of the second type (see 1.6) and differ essentially from definitions *per genus proximum et differentiam specificam* (see 1.7). However, operational definitions present certain difficulties for logical analysis. One gets the impression that Bridgman's definitions may be used to define a certain fixed length of some object rather than the abstract object—length in general.

The problem thus arises of defining physical magnitudes in their most general form (length in general, mass in general, force in general, etc.). This may be done by successively applying the identification abstraction to a certain sequence of concrete operational definitions and by applying the identification principles formulated above.

Suppose that we are interested only in the length of some objects, ignoring other characteristics of the same. Assume that we have to define operationally the lengths of the objects A, B, C, ... The Dfd's of these definitions will be respectively: 'The length of the object A', 'The length of the object B', 'The length of the object C', etc. Descriptions corresponding to the Dfn's of these definitions may differ from each other (1) by magnitudes of measurement results; (2) by instruments (and, consequenly, sets of expe-

rimental measuring procedures) used to define the objects' lengths, etc.

Let us now compile a table of the possible Dfn's in the definitions of the length of the objects given (see below). Each line of the table will contain the Dfn describing the various results obtained in applying the same instruments (and, consequently, identical experimental measuring operations). For instance, line (1) may contain the Dfn's describing the results of measuring the objects A, B, C... with the aid of a ruler, which may be 1 m, 2 m, 3 m. Line (2) may contain the Dfn's describing the results of measuring the objects A, B, C... with the aid of a tape-line, the results being the same as in the previous case. Every column of the table will then comprise the Dfn's describing the results of measuring *one and the same* object A, with the aid of *different* instruments (consequently, with the aid of different experimental measuring procedures). Thus the first column will register the object's length of 1 m measured by a tape-line, a theodolite, etc. The second column may register the object's length of 2 m measured by a ruler, a tape-line, a theodolite, etc.

	(1)	(2)	. . . . .	(n)
(1)	D_1^1	D_2^1	. . . . .	D_n^1
(2)	D_1^2	D_2^2	. . . . .	D_n^2
.	.	.	. . . . .	.
.	.	.	. . . . .	.
.	.	.	. . . . .	.
(k)	D_1^k	D_2^k	. . . . .	D_n^k

The letter D designates the Dfn. The identity of the superindices signifies the identity of the instruments, their difference, a difference in the instruments. The identity of the subindices signifies identity of the results of measurement, their difference, a difference in these results.

Let us now perform certain identity abstraction operations.

The common feature present in all the Dfd's, that is, A, B, C,... will apparently be length in general. The co-

lumns (1)-(n) of our table may apparently be identified (made indistinguishable from one another) according to the identification principle (I). The lines (1)-(k) may be identified (made indistinguishable from one another) according to the identification principle (II). That means that *all* the elements of our table (D_1^1)-(D_n^k) will be identified (made indistinguishable from one another) by the two identification principles.

This means that length in general may be represented by any Dfn of the table (including any sets of such Dfn's). The definition of length will therefore take the form of a proposition where the Dfn is a disjunction of all the elements of the table:

Length in general $\equiv D_1^1 \vee D_2^1 \vee D_1^2 \vee D_1^3 \vee D_2^2 \vee , \ldots, \vee D_n^k$.

Here are some examples of definitions of physical magnitudes: 'Length is something that may be measured by a tape-line', 'Time is something that is measured by clocks', 'Speed is something that is measured by an accelerometer', etc. The table does not include certain complex elements which do not satisfy the identification principle (III), that is, those which do not satisfy, for some reason or other, the procedure for reducing them to elementary entities. These elements form a new distinct set for which a new concept of length will hold (of this nature is the concept of optical length outside the Earth, or that of lengths comparable with the electron diameter).

4.3. ON THE TYPES AND STATUS OF OPERATIONAL DEFINITIONS

Following Bridgman, the concept of operational definition has been extended. We believe it expedient to distinguish between operational definitions in the narrow and broad sense.

The former should include (1) operational definitions in the sense of Bridgman, that is, definitions describing experimental measuring procedures specifying a certain physical magnitude; these are the basic definitions in physics; (2) operational definitions including computational operations along with experimental measuring procedures; (3) definitions describing specific experimental operations of discovering certain physical objects (phenomena, properties, relations). These definitions are used to study

objects qualitatively. They may be illustrated by the definition of acid through describing its effect on litmus.

It is advisable to include among the operational definitions in the broad sense the so-called *constitutive definitions*,[1] that is definitions of physical magnitudes based on mathematical formulas used within the framework of relatively complete theoretical constructions. Some of these formulas describe correlations between physical magnitudes which are called the laws of nature.

We know from logic that any explicit real definition of a meaningful theory of the form Dfd $\equiv$ Dfn (the terms of the definiendum and the definiens are equal in extent, that is, they are correlated with precisely the same objects; they have identical interpretations) may be transformed into an equivalent proposition Dfd $\sim$ Dfn establishing the equivalence of the terms Dfd and Dfn, or into the proposition Dfd=Dfn (see 3.5). In the latter case the proposition is to be read as follows: the objects satisfying the Dfd and the Dfn are identical objects. If a physical theory is rigorously constructed and certain primitive undefinable concepts (and the corresponding terms and objects) are singled out in it, some formulas with equation may be regarded as definitions.

Thus, if length (*cm*), time (*sec*), and mass (*g*) are regarded as primitive undefinable concepts in classical mechanics, its other concepts are introduced by definitions.

Outside the theory, that is, within classical physics in general, these primitive objects are defined at different levels through the relevant standards, the units of length, time, and mass. Length may be exemplified by an analogue of ostensive definition (see 1.11)—the standard metre preserved in Paris. Mass may be defined in a similar way, too. But the same mass beyond a closed theory may also be defined as the quantity of matter, at the level of methodological analysis. The same primitive objects may also be defined operationally (in the narrow sense of the term). Such operational definitions are usually introduced into a theory to complement constitutive definitions. They are an important means of linking theory and experimental measuring activity.

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[1] H. Margenau. *The Nature of Physical Reality*. McGraw-Hill Book Company, NewYork-Toronto-London, 1950, pp. 232-40.

These primitive objects may serve as the basis for constitutive introduction of other, more complicated objects into a closed theory, the theory of classical mechanics itself being constructed genetically. Thus speed is introduced by the formula $v=\frac{s}{t}$ (cm/sec); acceleration as speed alteration is introduced by the formula $a=\frac{v}{t}$ (cm/sec^2); force is introduced by the formula $f=m\cdot a$ $(g\cdot cm/sec^2)$, etc. Mathematical formulas perform a number of functions, including that of mathematical description of the laws of nature, of the means of computing the values of the relevant functions on the basis of the information obtained experimentally for the values of the corresponding (independent) variables. In these cases they are not definitions and yet may be regarded as definitions of physical magnitudes as well.

Formulas may be regarded as constitutive definitions of physical magnitudes only when the magnitude introduced by a formula is equated with magnitudes that have been defined in the system independent of the magnitude being defined. The magnitude defined will be definitially reduced to a combination of primitive physical magnitudes different from it, and its definition will not contain a vicious circle. Otherwise, definitional reduction will make the formula tautological.

Thus, within the framework of the theory considered here the formula $f=m\cdot a$ may be regarded as a constitutive definition of force. Introducing the entity which a equals by definition, we obtain $f=m\cdot\frac{v}{t}$. Then, substituting the magnitude which v equals by definition for v, we obtain $f=m\cdot\frac{s}{t\cdot t}$ $(g\cdot cm/sec^2)$. The concept of force has thus been expressed by a combination of primitive objects which may be defined independently from the concept of force even beyond the domain of science. However, the formula $a=\frac{f}{m}$ obtained from $f=m\cdot a$ cannot be regarded as the definition of acceleration. In trying to reduce it to combinations of primitive objects we shall obtain a tautology. Indeed, in substituting the expression equal to f $(f=m\cdot a)$ for f, we obtain $a=\frac{m\cdot a}{m}=a$. Thus the formula $a=\frac{f}{m}$ is a func-

tion we can use for certain computations, but it does not have the status of a definition.

There is no generally accepted view on operational definitions in the literature on the logic and methodology of sciences. Some authors (e.g., Przelenski and Bunge) do not regard operational definitions as definitions at all.[1] We believe that operational definitions are definitions in their own right (at any rate, they satisfy the definitions of definition given in 2.2). Just as we sometimes endeavour to replace conceptual and intuitive definitions in mathematics and logic by effective generative and decision procedures (e.g., inductive definitions of the natural number series and formulas in some calculus) having the status of definitions, we try to make decision procedures more effective in physics.

It is not enough for a physicist (at certain levels of experimental measuring activity) to know the meaning of the relevant terms and to be able to distinguish certain classes of physical objects (e.g., physical magnitudes) from others by describing their properties. In definitions of this kind the distinctive features are frequently no clearer intuitively than their definienda; besides, these features or properties are often not given in direct experience (e.g., the definition of force as the cause of motion, of mass as the quantity of matter). These definitions are similar to the Frege-Russell definition of number (see 1.8) in which the objects that are intuitively clearer and more amenable to verification are defined through more abstract and complicated ones. Such definitions are unconstructive and ineffective; they are hardly applicable in mathematical or physical activity proper. They may have some interest only at an interpretative level, at the level of philosophical and methodological theorising.

The activities of a physicist are primarily concerned with experimentation, observation, generalisation, and computation. What he therefore needs are the methods of identification and differentiation involving application of experimental measuring and computation procedures to the objects under study.

[1] See *Problemy logiki nauchnogo poznaniya* (Problems of the Logic of Scientific Knowledge). Nauka Publishers, Moscow, 1964, pp. 307-13; M. Bunge. 'Do Operational Definitions of the Concepts Exist in Physics?'. *Voprosy filosofii*, No. 11, 1966, pp. 66-77.

Operational definitions in the narrow sense comprise methods for the identification of physical magnitudes that also include descriptions of effective procedures for their discovery and quantitative evaluation.

If in solving some experimental problems we perform certain procedures, we are dealing with an abstract object like length; through these procedures we determine the length of the object in question. It may be objected that in devising an experiment and applying particular measuring instruments and procedures we had to know beforehand that we were dealing in this case precisely with length, and that means that we resorted to a definition of length independent of experimental measuring apparatus.

That is indeed so, but that is no argument against the existence of operational definitions, since it applies to most analytical definitions obtained at the level of experimental knowledge. Thus, to develop the scientific definition of man as a social being, we should be able to distinguish between man and other animals. This distinction is made at a level of knowledge dominated by intuition. In our everyday life we frequently resort to this sort of 'intuitive definitions' (cf., the ability to distinguish between the various kinds of mushrooms at this level of cognition; scientific descriptions do not guarantee such fast and accurate orientation in this case).

The concept of length was also formed in everyday activity long before its operational definition. It is quite possible that this abstract object was singled out through definitions by abstraction (as something that is equal in length to a certain standard) and a subsequent generalisation of lengths equalling different standards (see 1.8). The singling out of abstract objects at a certain level of knowledge does not interfere with their subsequent operational introduction in physics for obtaining better results in this field.

In the foundations of mathematics the natural number series is sometimes introduced by inductive definition. But it never occurs to anyone to doubt this definition on the ground that we know already what the natural number series is, that we can count without this definition, etc.

Every operational definition in the narrow sense may be interpreted, e.g., as a semantic one: the Dfd will here be a certain term and the Dfn, a description of the experi-

mental measuring procedures; it will thus have the form 'Dfd'≡Dfn.

As one and the same physical magnitude may be defined by different sets of experimental measuring operations (including the description of the results of their application), we may present the definitions of physical magnitudes (e.g., length, mass, force) as a disjunction (see p. 160):

$$'Dfd' \equiv Dfn_1 \vee Dfn_2 \vee Dfn_3 \vee \ldots .$$

It is assumed that we can identify Dfn_1, Dfn_2, Dfn_3, . . ., etc. with one another. These principles of identification were formulated above.

Any explicit meaningful nonoperational definition may, generally speaking, be represented in general form as a disjunction, too, if each of its members corresponds to one and the same set of objects coinciding with the set of Dfd's. We may, for example, define man as

(1) 'An animal capable of producing instruments of labour' (Dfn_1),

(2) 'An animal capable of developed articulate speech' (Dfn_2),

(3) 'A reasonable animal' (Dfn_3), etc.

The definition of man way be written as a proposition of the form

$$'Dfd' \equiv Dfn_1 \vee Dfn_2 \vee Dfn_3.$$

This definition may be read as follows: 'Man is an animal capable of producing instruments of labour, capable of developed articulate speech and an animal that is reasonable.' [1]

From the formal viewpoint this definition carries redundant information: the set of objects defined is unambiguously given by any of the members of the disjunction. Each of these three definitions is therefore formally (extensionally) equivalent to the others:

[1] If we know that one and the same set of objects corresponds to the Dfn_1, Dfn_2, Dfn_3, then the Dfn_1, Dfn_2, Dfn_3 may be connected by the sign of conjunction as well. The sum of the sets will be the same as their product. In our view, however, the disjunction sign is in better intuitive agreement with the conception of the predicates corresponding to the Dfn_1, Dfn_2 and Dfn_3 as independent ones.

$'Dfd' \equiv Dfn_1$,
$'Dfd' \equiv Dfn_2$,
$'Dfd' \equiv Dfn_3$.

Carnap also considers the problem of defining physical concepts.[1] He points out that there is no hard-and-fast line between empirical and theoretical laws. According to him, empirical laws differ from the theoretical ones in that they contain either terms that are directly observable or those that are measured by relatively simple techniques. Among the observable (empirical) terms he includes 'pressure', 'volume', 'temperature', 'mass', 'electric current', 'length', etc. The physical magnitudes appearing in the physical laws (e.g., the Ohm law, the Gay-Lussac law, etc.) are either observed directly or at most measured by simple procedures. Among the unobservable (theoretical) terms are included the terms correlated with microprocesses: 'electromagnetic wave oscillation', 'field', 'molecule', 'atom', 'electron', 'electron spin', etc.

Empirical and theoretical laws are supplemented by correspondence rules connecting the empirical and the theoretical terms. An example of this kind of rule is: 'Gas temperature is proportional to the mean kinetic energy of its molecules.'

Carnap seems to discount the fact that the same term may be regarded both as empirical and as theoretical. Thus, 'current intensity' may be defined as an empirical term (if this expression is defined operationally, namely, as a magnitude measured by an ammeter) and as a theoretical term, namely, as a characteristic of electric current introduced on a purely theoretical level, as the property of bodies manifested in the motion and interrelations of the microparticles. Speaking generally, there is a tendency in a number of sciences to introduce an empirical operational definition for the relevant theoretical one: the essence that is of significance for the construction and explanation of the world picture is linked with the phenomenon, which is directly and constructively registered by observation and measurement.

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[1] See R. Carnap. *Philosophical Foundations of Physics.* Basic Books, N. Y., 1966, Chapter 24.

Correspondence rules function therefore not only as the necessary means of obtaining empirical laws from the theoretical ones but also as operational definitions of the theoretical through the empirical.

4.4. OTHER KINDS OF DEFINITION USED IN THE MATHEMATICAL NATURAL SCIENCE

Let us now consider an example illustrating the rule for defining definitions as operational or nonoperational ones. The following definitions of force appear in the literature on physics and the methodology of physics: [1]

(1) force is the cause of motion;

(2) force is a push or a pull;

(3) force is the physical magnitude proportional to the extension of the spring in the spring balance;

(4) force is anything that causes a spring balance to be extended;

(5) force is anything that neutralises a weight; its measure is the amount of weight which it holds in equilibrium;

(6) force is an agency causing departure from straight-line, uniform motion in material bodies;

(7) force is mass times acceleration;

(8) force is the negative gradient of the potential energy.

This list could be extended.

Definition (1) at one time seemed satisfactory to scientists and was used not only in the philosophical but also in the special technical sense, viz. as the increment of the speed vector. The word 'cause' was in this case interpreted as a substitute for 'dynamic pressure'. The d'Alembert principle is based on this definition (in the interpretation given here). Definition (2) correlates the physical concept with our kinesthetic sensations. In definition (3) the anthropomorphism of definition (2) is overcome, the magnitude of force being defined operationally. Definition (4), as distinct from definitions (2) and (3), stresses the fact that force is a certain substance acting on the spring balance as a causal factor. In definition (5) force is defined with regard to weight, and an operational procedure is given for defining its magnitude as a function of weight, or gravitation mass. In definition (6) force is de-

[1] H. Margenau. *Op. cit.*, pp. 223-24.

fined through inertia and the concept of inert mass linked with it. In definition (7) force as a physical magnitude is defined by the relevant mathematical correlation corresponding to a law of nature. Definition (8) figures in a number of dynamic systems, where such magnitudes as the Lagrange function or the Hamilton function or potential energy appear to be intuitively simpler and more self-obvious.

Of the definitions cited here only (3) and (5) may be viewed as operational ones without reservations, according to Bridgman. In conformity with the broader definition of operational definitions suggested above, these will include not only definitions (3) and (5) but also (4) and (6), characteristic of the qualitative level of analysis, as well as (7) and (8), which are constitutive operational definitions. Definitions (1) and (2) are not operational.

We believe that physics cannot restrict itself exclusively to the operational modes of introducing concepts (even in the broad sense of the term). First, this cannot be done because physical theories contain constructs that cannot be formed on the basis of experimental measuring procedures or even computational operations. They are introduced by a *Gedankenexperiment* of a special type. Of this nature are the idealisations of material point, inertia, ideal gas, absolutely resilient body, etc.

Second, at the level of interpreting the experimental data certain system or model objects are introduced which play an enormous role in the development of physical theories (e.g., atoms and the various elementary particles). Many specialists studying the structure of physical theories are trying to reveal the multiformity of definitions used at the various levels of physical knowledge, in the domains of the various languages corresponding to these levels. For instance, Caws [1] distinguishes five groups of definitions that are preferred at the various levels of physical knowledge. In the initial stages of physical cognition we frequently resort to ostensive and semi-ostensive (in a defferent terminology, to denotative and semi-denotative) definitions or, to be more precise, to exemplary definitions (definitions through examples) (see 1.11).

[1] P. Caws. 'The Function of Definition in Science'. *The Philosophy of Science*. Vol. 26, No. 3, 1959.

The next level of physical knowledge requires formulation of other definitions with regard to those physical events which have been registered in definitions of the form indicated above at the previous level. This is done by the so-called descriptive or other types of definition, mostly definitions *per genus proximum et differentiam specificam.* The purpose of these definitions is to single out and describe unambiguously the observed physical events through their properties. It is also at this level that observable constructs are formed and described, measuring procedures elaborated and systems of measurement units developed. Application of measuring procedures to the observed constructs transforms the latter into physical magnitudes. Operational definitions in the narrow sense are also widely used here. The next level of cognition involves the development of logico-mathematical apparatus. New signs are introduced into science to record empirical laws in the language of mathematics. Operational definitions are widely used here too, as well as nonoperational nominal abbreviative definitions that introduce the meanings of new signs employed in the formulas.

Next we proceed from the empirical level of knowledge to the theoretical one (to constructing a theory). The latter level involves interpretation of the logico-mathematical calculus, the introduction of new theoretical constructs (resp. of new logico-mathematical terms), and the unification of physical knowledge into an integral system. Apart from operational definition in the narrow sense, wide use is made of definitions describing the *Gedankenexperiment* introducing idealisations, as well as of nonoperational definitions introducing such objects as atoms, elementary particles, etc.

Next, a logico-mathematical calculus is constructed which unifies our knowledge of the objects under study into a single system, in which a number of empirically obtained correlations may be deduced as logical corollaries of the fundamental propositions of the theory. To define the meanings of the signs used in the formulas, one often resorts here to constitutive operational definitions. Definitions frequently have to be employed to establish the physical meaning of the so-called quasi-mathematical objects (e.g., expressions like 'square of the time', 'strain tensor', which are verbal notations of the respective sign expres-

sions appearing in the mathematical apparatus of physics). These levels of knowledge form an intricate network of correlations and are linked through various channels with experience.

The problems of developing a language for the theory, of interpreting instrument readings and terms appearing at the level of mathematical description, become extremely involved in modern quantum physics, giving rise to fundamental difficulties. We can obtain knowledge about the object under study only from the instrument readings, the microobject itself never being immediately given to the observer. Moreover, the nature of the particle-wave duality of quantum mechanics is such that one class of instruments registers only the particle properties of the microobjects while the other, only their wave properties. These two kinds of information, on the one hand, mutually exclude each other, and on the other, they presuppose each other, as they are both necessary for a complete description of the microobject. Besides, the instrument itself, as well as the experimental data obtained with its aid, are described in classical terms, whereas the results of the experiment, the mathematical descriptions of the experimental data frequently have to be interpreted in nonclassical terms.

To explain the descriptions obtained researchers have to resort to a rather extraordinary procedure. In the classical theories essentially founded on interpretation of experimental data (e.g., in the kinetic theory of gases), as we know, explanation essentially hinges on a reduction of the complex to the simple: at the level of interpretation, objects are regarded within the framework of the laws of classical mechanics that are directly observable and verifiable. In quantum physics, however, scientists have to postulate the existence of certain hypothetical, sensually unperceived objects to explain the descriptions obtained, and, moreover, they have to endow them with properties, interactions, and motions that cannot be described within the customary classical framework.

Nonclassical concepts are usually described as uncommon combinations of elements having certain properties of the macroworld. This inclusion of classical concepts among nonclassical ones is inevitable, since the language of interpretation of nonclassical theories would otherwise be incomprehensible at all. Speaking of explication in this

case, we interpret it in a specific sense, as there is no question at all of unconditionally reducing the complex to the simple. Reducing explication to elaboration of a convenient and comprehensible *façon de parler* will apparently fail to satisfy most physicists.

Apart from the function of attributing a physical meaning to the phenomenological and mathematical descriptions (which is achieved by means of definitions), the interpretative language in microworld physics has another significant function as well. The point here is that the objects of microworld physics are far less stable and rigid than the objects of macroworld physics. This is first and foremost manifested in the fact that observation results (always mediated by an instrument) concerning elementary particles do not have the same definiteness and unambiguousness as the experimental results obtained in classical physics. The interpretative language based on unambiguous definitions of terms is an important means of their constructivisation. Margenau observes: 'In the case of atomic particles most observations scatter, and one achieves coherence by pulling them together in common reference to a system functioning, as it were, behind them.' [1] In his view, giving up the sphere of postulated interpretations (as was suggested by some positivists and operationalists) would mean sacrificing language, which is the most important means of systematising experience and the most natural procedure for ordering it. Having made this sacrifice, the physicist would have to elaborate a new schema 'for making sense out of the scattering welter of his observations',[2] which would apparently be even more complicated and devoid of the naturalness of the physical approaches to the analysis of experience which have justified themselves in all the physical theories.

Thus we see that physicists widely use definitions other than operational ones. These include a specific kind of ostensive (exemplary) definitions, definitions *per genus proximum et differentiam specificam* (as illustrated by specifying a newly discovered phenomenon), and genetic definitions (as illustrated by the introduction of idealised objects through a description of the modes of their formation).

[1] H. Margenau. *Op. cit.*, p. 228.
[2] *Ibid.*, p. 229.

These definitions are widely used in the most diverse spheres of knowledge.

A significant role is played in physics by definitions of the various kinds of model objects through descriptions which include, apart from a description of the objects' properties, also some data on their quantitative characteristics (e.g., definitions of the various elementary particles, types of radiation, etc.). We suggest calling them *synthetic quantitative definitions.*

The so-called definitions through law described by B. M. Kedrov seem to be specific for mathematical natural science.[1] These include several types. We shall here deal with just one of them. At the microworld level (or the level of interpreting the known macroscopic descriptions) ideal gas may be defined as a certain idealised object—e.g., in these terms: gas is called ideal if it has only the kinetic energy of moving particles interacting only in collisions, as if they were infinitely small bodies, the potential energy of particle interaction equalling zero. Kedrov defines the ideal gas concept at the phenomenological level through the laws which express the physical state of gas.

'This concept,' he writes, 'is based entirely on the Boyle-Mariotte and Gay-Lussac laws integrated in the Clapeyron equation expressing the physical state of gas. It was discovered that all the existing gases deviate in one direction or another from these laws approximating them with precision only under conditions of vacuum and a temperature least approaching critical temperature; so the need arose to form an abstract conception of gas that would take into account the ideal case of gases satisfying the above laws and equations without any deviations... This is an example of definition of a concept through law (or a number of interconnected laws), as the ideal gas concept has no meaning other than precise conformity with the laws of gases outlined above: ideal gas is that kind of gas which conforms with these laws.' [2]

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[1] See B. M. Kedrov, 'Definition of Scientific Concepts Through Law', *Dialektika i logika. Formy myshleniya* (Dialectics and Logic. Forms of Thought), Akademiya Nauk SSSR Publishers, Moscow, 1962, pp. 66-112.

[2] *Ibid.*, pp. 68-69.

4.5. DEFINITIONS AND EXPERIENCE

If one considers definitions in experimental sciences in isolation from the context of science (we do not mean definitions of units of measurement), they will appear as experimental facts in which only the choice of the Dfd term is conditional (if the definition is synthetic, see 1.3). However, if definitions are to be considered in the context of science as a whole, of some scientific theory or other, the problem of correlation in definitions of experimental knowledge and knowledge introduced by the conventional elements becomes nontrivial. Mandelshtam has expressed some extremely interesting views bearing on this point.[1]

Application of the fundamental physical magnitudes is based on selecting a standard of measurement and defining procedures performed on this standard. Definitions of the units of measurement are close to ostensive ones (see 1.11). They form the basis for other definitions with which a physicist operates, serving as a link between mathematics, physics and the real material world. 'First [Mandelshtam writes], we must have a *standard*, a unit of measurement. For that, we should be able to demonstrate some real object. We should be able to build a bridge between concepts manipulated by mathematics and the real objects. To achieve this, we should be able to demonstrate a real object, for instance, the rod that is preserved in Paris and that is regarded as a unit. We should then be able to define what it is *to measure*, that is, what we mean by propositions "the rod that is here and the rod that is in some other place are equal". We have arrived at the following conclusion: if the rod there and the rod here coincide with the standard metre, the latter being first applied to the rod here and then taken to the other place where we again observe a coincidence, then the rods are one metre long.' [2]

As for the fact that the standard rod equals one metre, it is not an experimental fact. This fact is determined by the definition accepted (this rod is what it is by condition, by agreement). The statement of a certain rod being equal to 5 *m* is already an experimental fact (knowledge attained

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[1] See L. I. Mandelshtam. *Polnoe sobranie trudov* (Complete Works), Vol. V, Akademiya Nauk SSSR Publishers, Moscow, 1950 pp. 177-87.
[2] *Ibid.*, p. 182.

experimentally) in the sense that it registers the results of measurement which prove that the length of the given rod is five times greater than the length of the standard rod. On the other hand, the statement '1 *m* equals 100 cm' is not experimental.

Other standards introduced by definitions may also be used to measure length (e.g., the wave length of cadmium light). It may so happen that the standard lengths thus introduced are equal. This fact represents experimental knowledge. However, the standard measures introduced must satisfy some factual requirements: they should be rigid, unalterable in the process of transference, convenient, practicable, etc. Thus, on the one hand, definition (of standards—*D.G.*) involves a measure of arbitrariness, and on the other hand, it is fruitful, as it is permitted by nature in the sense that it satisfies the requirements of unambiguousness, invariability, etc.[1]

In physics, one should distinguish between propositions that are true factually (by virtue of their correspondence to the object studied experimentally) and those that are analytically true. Thus, we may regard the Earth's rotation as a clock, considering time intervals corresponding to equal rotation angles of the Earth as equal (this is a definition). Then the statement that 'The angles of the Earth's rotation in equal time intervals are equal' will be an analytic truth. In a system of knowledge where the above operational definition of time is accepted, the question, 'Is the Earth's rotation really uniform?' will be regarded as meaningless. It is impossible to answer it since our clock's definition is based on the concept of the Earth's rotation. However, it is quite meaningful to ask by what *t*-function the process of pendulum swing will be expressed.

In a similar way, time may be defined from the Newton laws if the latter are considered valid. 'For example, the clock is represented by the body's inertial motion and it is postulated that equal time intervals correspond to equal paths. In this case, however, it is meaningless to verify whether the motion of the body is uniform.' [2]

Definitions of standards and other concepts must necessarily satisfy the requirement of expediency and rea-

[1] L. I. Mandelshtam. *Op. cit.*, Vol. V, p. 182.
[2] *Ibid.*, p. 180.

sonableness, which is based on the sum total of mankind's experience. Generally speaking, man could be defined in such a way that the class of men would include apes, too. This definition, however, would be unreasonable, since many statements about man that are experimentally true would appear to be false, many norms and regulations would prove to be unrealisable, and the set of men in the generally accepted sense would have to be given a new name by definition just the same, as the study of man in the generally accepted sense is practically important for the society and each of the elements of the class of men possesses peculiarities that are characteristic of just these elements. Although 'nature does not dictate definitions that are absolutely binding, neither does it permit any sort of definitions'.[1]

Of considerable methodological interest in this respect is Mandelshtam's analysis of two examples of definition. The first concerns the existence of the antipodes, the second, the definition of simultaneity.

When we walk on our feet, we hardly need a special definition of what it is to walk on our feet. But, if someone insisted on such a definition, we could formulate it as follows: '*N* walks on his feet then and only then when his head is farther away from the earth than his feet, and anyone walking with his feet up is antiparallel in relation to *N*.' The second distinctive feature (the feature of antiparallelism) indicated in the Dfn is perhaps redundant, but it does not spoil the definition. However, when the thesis that the Earth is a globe came to be rejected on the ground that it entails the existence of men who walk with their heads down and the feet up (the problem of the antipodes), this definition had to be revised. The antipodes are indeed antiparallel in relation to any person *N* walking on his feet (whose head is farther from the earth than his feet). This definition could no longer serve to refute the existence of the antipodes, as the first part of the Dfn entails the nonexistence of the antipodes while the second part entails their existence. To correct the definition, we should delete the second part of the Dfn although it does describe experimental facts (at any rate, the facts in the experience of the person who introduced the definition). The corrected definition which may be used to prove that the antipodes

[1] *Ibid.*, p. 179.

do not exist will read: '*N* walks on his feet if and only if his head is farther away from the earth than his feet.' The feature of antiparallelism should not be included in the definition: the question of whether persons walking with their feet down or their feet up (for instance, at the circus) are parallel or not is a question of experimental research. 'It was assumed that "man walks on his feet" was a God-given conception, while the position of his head relative to the earth was just one of its features; it was believed that the two features *had to* go together. This prejudice was based on a failure to understand what it was that had to be defined . . . whether this will coincide with another feature or not is a question of experience.' [1]

It is sometimes very difficult to decide what belongs to definition and what to experience. Let us consider a second example touching on this problem.

The statement that something occurs in such and such a place and at such and such a time simultaneously is simple enough and does not require a special definition. Assume, however, that we send a signal from point *A* to point *B* separated from *A* by a certain fixed distance (say, 1,000 *m*) and that the motion is in a straight line. Assume further that it is possible to synchronise the clocks at *A* and *B* (we may, for instance, transmit a signal from *A* every 10 minutes, and if the clock at *B* shows that the signal comes every ten minutes we shall say that the clocks are synchronous). However, to determine the speed of a signal, bullet or shell, that is not enough. 'I know, for instance, that the signal was sent from *A* at 12 o'clock and received at *B* at 3 o'clock. That does not mean at all that I have to divide 1,000 by 3. That would be so if I were completely certain that at the moment when the signal was sent from point *A*, the clock at point *B* also said 12, just as the clock at point *A*.

'How can I learn this in actual fact? I can say that the clocks are synchronised if they point at 12 simultaneously at points *A* and *B* and, further, if they keep the same time. But how are we to know that two clocks placed at different points in space are synchronised or, which is the same thing, how are we to know that two events at different points in space, say on the Earth and on the Moon, take

[1] L. I. Mandelshtam. *Op. cit.*, Vol. V, p. 184.

place simultaneously? Einstein gave this answer: you cannot know it, you must give a definition of what it means that clocks at two different points keep the same time, that two events occurred simultaneously at two such points. One must define what simultaneity means for two different points separated from each other by any (or sufficiently large) distance but not coinciding with each other. There is no other procedure.' [1]

The concept of simultaneity proves to be one introduced by a definition. We introduce it by a procedure similar to those that were applied in introducing length and time.

Analysis of the problem of correlation of knowledge directly available from experience and those of its components that are included in definition by convention, demonstrates the importance of studying the correlation between the analytic and the synthetic, between *a priori* knowledge (given prior to any concrete forms of generalised experience) and *a posteriori* knowledge in the theories of mathematical natural science.

[1] *Ibid.*, p. 186.

CHAPTER 5

DEFINITIONS IN THE SOCIAL SCIENCES

5.1. SOME FEATURES OF SOCIAL PHENOMENA DETERMINING THE SPECIFICITY OF DEFINITIONS USED IN THE SOCIAL SCIENCES

We shall not consider those definitions in civil history which are used to specify historical events and facts. Such specification necessarily presupposes registering the spatial and temporal characteristics of the events being studied. When analysing, interpreting and specifying historical events, a historian uses general concepts, laws, principles, conceptions that do not, strictly speaking, belong to certain concrete civil histories but are formed in such sciences as political economy, philosophy, law, etc. On the one hand, historians use the concepts of these sciences to generalise and interpret factual material, and on the other, history itself serves as the basis for the formation of the fundamental concepts of social theories (cf. the formation and development of the concept of proletarian dictatorship by the classics of Marxism-Leninism on the basis of analysing the experiences of the revolutions in the capitalist epoch). Historians themselves play an important role in the formation of such concepts.

Assume that we have succeeded in separating the civil histories of the various peoples and countries covering different periods from the fundamental methodological part applied in concrete historical studies, in the analysis and interpretation of contemporary events, in solving concrete social and economic tasks of the society at each stage in its development.

What are the specific features of social phenomena and respectively of the social sciences determining the peculiarities of the definitions used in the social sciences?

1. Social change and development proceed at a faster rate than development in animate nature that is studied

by the biological disciplines (to say nothing of development in inanimate nature).

When defining the concepts, laws, and principles in the domain of inanimate nature, studied by such fundamental sciences as physics and chemistry, we disregard historical time.[1]

That means that the definitions and laws of these sciences do not include parameters indicating the historical periods in the development of inanimate nature in which a certain physical law is valid or for which the definitions of concepts introduced are true. It is assumed, for instance, that the gravitation laws, the Archimedes law, etc., are valid at present, were valid in the past and will be valid in the future whatever the extent of that future. This assumption is justified by sciences having no evidence to assert that the world has at any time been subject or will ever be subject to other physical laws.

However, in the geological and geographical sciences the formulations of a number of statements and definitions must already include the historical time parameter indicating that a certain state of things existed at a certain

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[1] Every material process is characterised by certain duration. The so-called astronomical time is used to measure the duration of various qualitative processes. Time is measured with the aid of an astronomic clock, that is, of some periodic process which may be repeated many times with identical duration. Duration together with a chosen mode of measuring it lies at the basis of an *abstract quantitative* concept of time according to which processes differing in duration are characterised from the quantitative point of view only: we use measuring procedures to establish whether any two processes are equal in duration or whether one of them is longer than the other. This abstract quantitative time is used in situations where qualitative characteristics are inessential in the sense that registering them in the course of a certain process does not transform this process into a unique stage in the general line of development (e.g., when the process being measured may easily be reproduced under the same conditions).

However, when we deal with a sequence of events or processes that are fundamentally irreproducible and unique, their uniqueness being essential, we link the abstract quantitative concept of time with a description of concrete qualitative characteristics of the process or event being measured: they are included in the temporal description. Processes having the same duration will in this case prove to be different in their concrete qualitative characteristics, their importance and significance, for instance, in the life of a certain person, people or mankind in general. We shall say that we are dealing with the concept of *historical time* in all such cases.

geological era, period or epoch whose duration is usually measured by millions of years.

Formulations of laws of society's development, as well as laws revealing correlations of its structural components, always necessarily include the historical time parameter characterising the period and the epoch with regard to which the laws are valid and the concepts introduced are applicable. It is therefore essential to distinguish between the various scopes of generality of social laws and concepts in terms of their validity on the diachronic (or genetic) plane.

2. The boundaries between many objects significant for the social sciences prove to be extremely dialectic and uncertain at the descriptive level, much more so than in inanimate nature (if the objects are viewed at the level of descriptive analysis). Introduction of hard and fast constructivist boundaries between them (usually crowned with elaboration of relevant definitions) always involves a greater measure of rule-of-thumb schematisation than is the case in phenomenological descriptions of natural phenomena. The task of introducing into science of such objects and corresponding concepts that would reflect the essence of the matter and at the same time be constructive enough is therefore extremely complicated in the study of social phenomena (see 6.3).

Practical application of any theory of mathematical natural science, as we know, always yields only approximate results, although the theory itself, operating as it does on abstract and idealised objects, can be a model of exactness and rigorousness. This feature is even more clearly manifested in the social theories, and this is only understandable as modelling social phenomena involves an even greater degree of rule-of-thumb schematisation (in most cases, anyway) than in exact natural theories.

3. Structurally, society is more complicated than any other sphere of reality. Indeed, in any given country at any given stage in its historical development it is possible to distinguish a great number of different levels of social life, its cross sections, aspects, and layers. Their mutual relations and interactions are extremely complicated. Of this nature is material production as a whole, including social production relations, social order, state structure, the forms of social consciousness, the forms of education

and upbringing, etc. These cross sections, levels and layers are in their turn multistructural formations whose components are subordinated to one another. A modelling of social life permitting reliable prognostication is impossible without revealing these structures and their correlations.

To take an example, natural language is only one of the components of social life, and its elementary objects, the signs, are incomparably simpler than the objects they code. Recent investigations have shown, however, that the structure of language is immensely complex and multilevel, and that great difficulties arise when one attempts to obtain new results even in modelling some of its individual aspects.

4. In class antagonistic society, the opposing classes pursue different interests. Opposing economic interests of the antagonistic classes are expressed in theoretical, ideological, and political struggle. The same events are often differently evaluated by the representatives of different classes, the same social objects (be it newspapers or weapons) are used to attain different, sometimes opposing, ends.

5. Laws characteristic of a given socio-economic formation are variously manifested in different countries both at the same time and at different times. The relevant general theory must be essentially based on the results of analysing those countries where the laws of the formation are represented in their most characteristic, classical forms.

The above may in a sense be generalised in this way: the objects of the domains studied by the social theories are incomparably more complex than the objects of the domains studied by other scientific theories. This entails complexity of propositions describing the domain under study and complexity of the methods applied here as well as of the definitions introduced.

The objects of the social sciences have many parameters. Thus, if the subject matter of a study is economic relations, they may be investigated in the context of a certain socio-economic formation (and we shall have to take into account this parameter, among others). It is also important to know whose class interests a scholar represents (particularly as far as conclusions and estimates are concerned). That means that we have to take into account this parameter, too. In applying a certain general economic theory,

we have to bear in mind the peculiarities of the historical development of the country to which the theory is applied, and introduce the appropriate adjustments. That means that to understand a definition formulated in some social science one requires contexts of varying length and analysis of some additional information which is not explicitly included either in the definitions or even in the content of the relevant theory.

Let us now consider the effect of the specific features of social reality (see points 1-5, pp. 178-82) and, correspondingly, of the social sciences, on formulation of definitions.

5.2. SPECIFICITY OF DEFINITIONS IN THE SOCIAL SCIENCES AND DIFFICULTIES OF THEIR FORMULATION

1. Formal logic usually considers concepts regardless of historical time. The extensional characteristics of concepts include as their elements those objects that exist at present as well as those that existed in the past or will exist in the future. It is in this sense that we speak, for instance, of the extent of the concepts plant, city, man, etc. In singling out subsets of such concepts, we usually do not take into account historical time either and regard them as simultaneously existing. Thus formal logic is indifferent to the temporal and genetic connections between such subsets of the extent of the plant concept as 'Palaeozoic plants', 'Mesozoic plants', etc. And this is quite understandable since the primary task of formal logic is the construction of a general theory of deductive inference, where the temporal and genetic distinctions between the subsets do not play an essential role.

These distinctions are frequently left unspecified in scientific studies owing to other considerations. They may prove to be irrelevant for the solution of scientific tasks other than the task of developing a theory of deductive inference (e.g., when we want to find out the specific general features of all plants irrespective of whether they exist at present or not and irrespective of their genetic connections). These genetic distinctions are frequently even disregarded as the science makes no definite reference to their existence (e.g., when we deal with the extent of the concept

of chemical elements, or with the subsets of the set of chemical elements).

However, in studying those domains of reality to which the historical time parameter may be applied, the need frequently arises to take into account these distinctions: to solve this task, it is important to distinguish between the *nebeneinander* (the ordering of elements side by side with each other) and the *nacheinander* (the ordering of elements one after another). One has to deal with these cases in any diachronic (including dialectic) analysis of reality. These distinctions play an enormous role in the study of social phenomena.

The scholar wishing to reveal the interrelations between the various stages or diachronic cross sections of social development, wishing to bring out the regular tendencies in the development of society and formulate some forecasts as to its future, always faces a number of difficulties arising in modelling the process of social development, in analysing it into component parts and in introducing the relevant concepts.

To explain this, let us take an intentionally simplified example. Suppose that we want to introduce by definition the concept of *modern* Russian vocabulary, that is, give a definition of this concept on a synchronic plane. The question immediately arises as to what should be regarded as Modern Russian. Assume now that the linguists have solved this problem. Let us call the historical period in which the language in its modern form functions period I. The vocabulary of the modern Russian language will apparently include all the words having lexical meanings and occurring in the relevant texts and defining dictionaries compiled within period I. The extent of this concept will comprise the union of the classes of words having a lexical meaning and occurring in the various dictionaries of period I.

At the same level we can form the concepts of the various subsets of the given set (e.g., of the different parts of speech having lexical meaning). It is known, however, that no synchronic cross section, strictly speaking, exists in the case of continually changing phenomena to describe and analyse which the concept of historical time is introduced: such a cross section always involves determining an interval of time in which an object may be viewed as invariable. And we do make such abstractions and reservations, we do

unify the synchronic and diachronic aspects of study. That means that in singling out synchronic cross sections we have to resort to essential simplifications where the domain of reality under study is represented in a typified, averaged and schematised form.

The same will have to be done in forming the conception of modern Russian vocabulary, since the vocabulary changes continually, with new words appearing in it and the lexical meanings of old words undergoing essential changes during period I.

Let us now assume that we have to introduce a general concept of the Russian vocabulary. This may be done in two ways: first, disregarding the factor of historical time and, second, taking this parameter into account.

Assume that we have devided the history of Russian into three periods: period I (the modern Russian language), period II and period III. Disregarding historical time, we represent the extent of the concept of Russian vocabulary as a union of the sets of words having lexical meanings used in all the three periods (they might be registered either in defining dictionaries or in the written sources). We are dealing here with the same case as in the example considered earlier, where we singled out the various subsets of the set of plants ignoring historical time. In this approach all the words included in the extent of the concept 'the Russian vocabulary' are viewed as existing *nebeneinander*. No properly historical analysis of the object (dialectical analysis included) is possible within this framework; for example, we cannot trace the tendencies in the development of the Russian vocabulary or make predictions about its future.

To introduce this concept on the basis of the diachronic approach, we must include the subsets corresponding to the periods I, II, and III in the extent of this concept as its elements, and these subsets must be thought of as a *series* of objects *ordered* on the basis of the historical time parameter. This is the necessary condition for analysing the object under discussion in the historical and dialectical aspects.

When transitions from one historical period to another involve essential changes in the qualities of objects, including leaps of a revolutionary nature, difficulties arise in defining concepts spanning periods that differ essentially from one another.

Let us consider in this light the definition of the concept of social class. A classical definition of this concept was given by Lenin: 'Classes are large groups of people differing from each other by the place they occupy in a historically determined system of social production, by their relation (in most cases fixed and formulated in law) to the means of production, by their role in the social organisation of labour, and, consequently, by the dimensions of the share of social wealth of which they dispose and the mode of acquiring it. Classes are groups of people one of which can appropriate the labour of another owing to the different places they occupy in a definite system of social economy.' [1]

This is actually a definition of *class in an antagonistic society*. It cannot be applied to classes in socialist society without proper correctives.

In the history of social development the situation is not infrequent when the definition of some social phenomenon formulated with regard to a certain historical period (e.g., one or several socio-economic formations) proves to be inapplicable to the next historical period (e.g., the following socio-economic formation). The original definition in such a situation has to be *generalised* and *modified*. Considerable methodological difficulties have to be overcome in the process.

Let us turn again to the definition of class. In the present work we cannot of course consider the very essence of the problem, that is, we do not intend to offer our own definitions of class. We shall restrict ourselves to a brief outline of the methodological points that have to be taken into account by the scholar elaborating a general theory of classes and class struggle.

The classes in socialist society, as we know, are nonantagonistic and friendly. To generalise Lenin's concept of class, to elaborate a *generalised definition* of class, it is necessary to find the *common features* of classes in an antagonistic and a nonantagonistic, socialist society. If such common features are nonexistent, we cannot call two different objects by one and the same name, as that would contravene the rules regulating introduction of terms in all the sciences. If there are no features common to both antago-

[1] V. I. Lenin. 'A Great Beginning'. *Collected Works*, Vol. 29, Progress Publishers, Moscow, 1965, p. 421.

nistic classes and the classes of socialist society, there would be no grounds for considering the workers and peasants of socialist society as belonging to different social classes—moreover, there would be no grounds for calling them classes.

We believe that Lenin's definition identifies all the essential features that have to be taken into account in solving the problem of whether to attribute class status to certain social communities, be they antagonistic or not. Since analysis of the object in question is conducted within the framework of an integral theory, the Marxist-Leninist theory of classes and class struggle, evaluation and definition of the new classes of socialist society must proceed, *mutatis mutandis*, from the same distinctive features that are indicated in Lenin's definition.

The processes of *generalisation* in the abstract and generalisation of definitions in particular have not been studied thoroughly enough with reference to social development in the methodology of science. We can single out at least two different kinds of generalisation here:

(1) Assume that there are phenomena $A_1, A_2, \ldots, A_n$ connected with one another genetically and evolutionally and forming a continuous series such that each of the events A_i generates the event A_{i+1}. It is apparent then that the common features of $A_1, A_2, \ldots, A_n$ will always be *some* modified properties of A_1 (these may also be the specific properties of A_1).

Let us consider an example of generalising the various forms of reflection at the level of living matter.

Reflection at the *pre-psychical biological level* (A_1) is reproduction in cell protoplasm and registering or coding in hereditary mechanisms, in the course of evolution, of the similarities and identities in the environmental components and the rhythmical changes in the environmental states, which facilitates the organism's adjustment to environmental conditions, its survivability, and formation of goal-directed behaviour. Absense of the nervous system does not preclude signalling activity in living organisms: it proceeds on the basis of physico-chemical processes going on in cell protoplasm. Already at this level, reflection is of anticipatory nature: the organism reacts expediently to the external factors that are significant for it, preparing itself for the ongoing cyclic changes in the environment

(e.g., plants preparing for wintering at the first signs of the cold coming).

Reflection at the *biological psychical* (conscious) level (A_2) is reproduction of directly acting environmental components and the periodic alterations of environmental states in the form of subjectively experienced images (e.g., sensations and perceptions) whose adequacy is confirmed by the expediency of the animal's behaviour. At this level the animal's ability for environmental orientation grows extraordinarily, being based on the realisation of detrimental or advantageous effect upon it of the various agencies experienced by the animal respectively as suffering (pain, fear) or pleasure. There is also an increase in the animal's ability to acquire personal experience on the basis of conditioned reflex activity which is a function of the nervous system.

Reflection at the *level of cognition* (A_3) is reproduction of the thing in its objective properties and in their mutual relations (irrespective of whether the latter are directly significant or irrelevant for the preservation of the subject's life), the reproduction being based on:

(a) practical activity and collective (not only individual) experience coded by means of artificially created sign systems and assimilated in the course of education and upbringing;

(b) application of procedures of cognitive activity ensuring virtually unlimited possibilities for the anticipatory reflection of reality.

Information about the world acquired at this level of reflection usually admits an interpretation that is amenable to truth-value assessments and to verification by direct or indirect juxtaposition with reality.

Having singled out the features that are common to all the forms of reflection listed above (that is, having applied to them an identification abstraction), we evolve in this way a general conception of reflection at the level of living organisms, that is, we generalise the concepts A_1, A_2 and A_3. Formulating explicitly the general and the specific in reflection at the level of animate nature, we shall thereby construct its definition.

What is common to all the forms of reflection listed above? The necessary analysis reveals the following characteristics common to A_1, A_2 and A_3. In all the forms of reflection

reproduction of reality takes place (although in different variants, of course); at all levels reflection is anticipatory; the process of reflection is conditioned (be it merely at some of its stages) on *contact* between the reflecting organism and the reflected object and on using *codes* as semiotic systems. These characteristics, however, in their elementary variants are precisely the characteristics of A_1. All the other characteristics superimposed on A_1 in A_2 and A_3 are eliminated in the process of generalisation as they are not common to all three, while the characteristics of A_1 listed above are inherited in a modified form both by A_2 and A_3.

The definition of reflection at the level of *living matter* may therefore be formulated as follows: *reflection at the level of living matter is anticipatory reproduction of reality by an organism, conditioned on direct contact between the organism and material objects at some stages of its formation and realised on the basis of semiotic code systems.*

The correlation of reflection forms is in a certain sense analogous to the correlation of forms of the motion of matter. The simpler, more primitive forms of reflection are not cancelled by the higher and more complex ones. Thus reflection as human cognition does not cancel the biological forms of reflection. Higher forms of reflection cannot be reduced to the more primitive forms, since the functions of the former cannot be performed by the latter. The higher forms of reflection, coexisting with the more primitive ones in the organism, dominate them.

What is common to all the forms of the motion of matter? It is, of course, the properties of the mechanical form of motion.

If the *stages* $A_1, A_2, \ldots, A_n$ (where $n \geqslant 2$) are registered *in the development of one and the same phenomenon* and if the correlation is satisfied that, given any A_i and A_{i+1}, their common features are the modified features of A_1, this correlation is, in our view, one of the characteristic features of any evolutionary process (including historical processes).

(2) In a *revolutionary* transition from one socio-economic formation to another some properties of the previous formation are usually eliminated entirely, while other properties undergo essential, radical transformations and modifications. Thus, in the transition from capitalism to socialism exploitation of man by man disappears, private property is abolished, while commodity production, state, law, etc.,

remain, although they are altered to such a degree that they may be said to have a different essence. Essential differences are revealed in the definitions of the bourgeois state and the Soviet state, bourgeois law and socialist law, etc. The very fact, however, that in all such cases we are dealing with characteristics of state, law, etc., indicates that these social phenomena have similar features as well.

Marx, Engels, and Lenin have elaborated classical definitions of state, law and other social phenomena as applied to the antagonistic class formations. In elaborating a general theory of social development (which is the only possible and justified framework for comparing social phenomena of different historical epochs) it is natural to try to elaborate generalised conceptions of state, law and other social phenomena and give their definitions.

We have shown that for the evolutionary process the generalised concept and its definition are based on some *previous* states of the evolving phenomenon; however, the generalised concept (and the corresponding definition) of phenomena included in revolutionary transformations may *in some cases* be based on their *subsequent* (historically more progressive) states.

Thus, in elaborating a generalised definition of classes (both antagonistic and nonantagonistic, characteristic of socialist society) only some of the specific features indicated in Lenin's definition will be included in its Dfn (e.g., the Dfn will not include the feature that 'classes are groups of people one of which can appropriate the labour of another'). Some of them will be modified in accordance with the modifications in their nature under socialism while other will remain unchanged (e.g., their differences stemming from their participation in the system of social production and organisation of labour).

The extent of such social concepts comprises objects ordered with regard to historical time. When some phenomenon emerges in the process of social development through evolution, it is difficult to draw a constructivist boundary determining the time of its emergence. This makes it more difficult to use the relevant definitions (cf. the definition of the concept of imperialism). Lenin wrote in this connection: 'Needless to say, of course, all boundaries in nature and in society are conventional and changeable, and it would be absurd to argue, for example, about the particular

year or decade in which imperialism "definitely" became established.' [1]

'The most reliable thing in a question of social science and one that is most necessary in order really to acquire the habit of approaching this question correctly and not allowing oneself to get lost in the mass of detail or in the immense variety of conflicting opinions—the most important thing if one is to approach this question scientifically is not to forget the underlying historical connection, to examine every question from the standpoint of how the given phenomenon arose in history and what were the principal stages in its development, and, from the standpoint of its development, to examine what it has become today.'[2]

We run into difficulties of establishing precise, hard-and-fast boundaries in the social phenomena even in the simplest cases. When the question arises, for instance, at what age people should be given the right to vote, a definite age limit is established (18 years in the USSR).

This limit is in a certain sense formal. First, it is formal because it does not take into account the fact that some persons may be mature enough to vote both before and after 18. Second, it is formal because it permits a simple solution to whether or not a person may vote by the purely formal age criterion, all other conditions being equal. It is easy to imagine what practical difficulties might arise if every person between the ages of, say, 16 and 20 or 25 had to be specially examined for the purpose of determining whether or not he or she is eligible to vote. Besides, it would apparently be neither democratic nor humane. This age limit is at the same time *rational*, since experience shows that by this age a person is on the whole mature as a citizen and may participate in election.

Reference to an age limit forms part of the relevant definition (e.g., the definition of universal suffrage). Universal suffrage is the right of all the citizens of the USSR having reached the age of 18 to vote, irrespective of their race and nationality, sex, religion, education, residence, social background, property status, and past activities, with the exception of those who have been legally certified

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[1] V. I. Lenin. 'Imperialism, the Highest Stage of Capitalism'. *Collected Works*, Vol. 22, Progress Publishers, Moscow, 1977, p. 267.
[2] V. I. Lenin. 'The State'. *Collected Works*, Vol. 29, p. 473.

insane and persons convicted in court and legally deprived of electoral rights.

Similar difficulties arise in settling the problems of old-age pensions, school-going age, etc.

Similar difficulties crop up in elaborating definitions of ethics. Thus, the basic categories of ethics, good and evil, are usually defined in a nonconstructive way. The *Philosophical Encyclopedia* says: 'Good and evil are the basic categories of ethics used in moral evaluations of the various phenomena, actions, the motives of the activities of men, etc. Good is an ethical category denoting the sum total of the conditions of life as well as ethical acts, principles and norms of behaviour positively appraised by classes, societies, and individuals. Evil is a category denoting negative phenomena in the individual's social and personal life, subject to moral censure and condemnation.' [1]

The Dfn's of these definitions are extremely vague, particularly the terms 'positive' and 'negative'. These terms are defined in logic as incomplete: they assume different meanings depending on their use in context.

Elaborating a scientific definition of good and evil involves real difficulties not only because these categories have changed their content at different historical periods but also because at every given level of social development any social phenomenon is contradictory, appearing as a multitude of links, mediations and interactions which reveal its negative and positive aspects. It is therefore often a question of the proportion of the positive and negative in the phenomena, including those allowing such moral evaluations. Good and evil are therefore such in definite respects only, just as medicines taken by a sick man are beneficient in one respect and harmful in another. It is all a matter of evaluating the proportion of good and evil in a certain social phenomenon regarded in relation to goals whose realisation is in the interests of the entire society and its progress.

Some social theories include measures for abstract objects which thus begin to play the role of magnitudes similar to physical magnitudes. One usually resorts to statis-

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[1] *Filosofskaya entsiklopediya* (The Philosophical Encyclopedia). Vol. 2, Moscow, 1962, p. 27.

tically mean parameters here. Such is Marx's definition of the socially necessary labour-time: 'The labour-time socially necessary is that required to produce an article under the normal conditions of production, and with the average degree of skill and intensity prevalent at the time.' [1]

At each stage in the development of production, evaluation of socially necessary labour-time thus presupposes operating with some mean parameters (mean level of skill and intensity of labour), and production conditions that are typical (normal) for the given time and the given country. The natural sciences, too, resort to this sort of mean quantities and statistical procedures. The mean indices in the field of social phenomena are extremely variable and unstable (at any rate, much more variable and unstable than in inanimate nature and technology). They change constantly in the process of social development.

(For a more fundamental analysis of difficulties involved in the elaboration of definitions due to the complexities of drawing constructivising boundaries in the domain of social phenomena see 6.3.)

The complexity of social life, the existence of numerous aspects and layers in it also influence the nature of definitions in the social sciences. Let us point out two relevant factors here.

There are two kinds of definitions in the theories of mathematical natural science. First, there are the definitions in which the Dfd and the Dfn are described in a language of the same level. Of this nature are operational definitions in Bridgman's sense (see 4.3), definitions through indication of the properties of objects at the level of phenomenological description, constitutive definitions (see 4.3) etc. Let us call these definitions *homogeneous*.

Homogeneous definitions are illustrated by the operational definitions in 4.3 of physical magnitudes, acid, the constitutive definition of mass, as well as definitions of gases, liquids, and chemical elements through a description of their properties.

As for Newton's definition of mass (see 4.1), the defini-

[1] K. Marx. *Capital*. Vol. I. Progress Publishers, Moscow, 1977, p. 47.

tions of chemical elements by their atomic weight, the definition of force by cause of motion, these are *heterogeneous* definitions; here the Dfd and the Dfn are described in *languages of different levels*: e.g., the Dfd is characterised at the level of descriptive language while the Dfn, in the language of explication (or interpretation).

Many homogeneous definitions in the social sciences are formulated in the language of phenomenological description. Homogeneous definitions continually appear in Marx's *Capital* at the theoretical level: the language of theoretical description and the language of interpreting the results of theoretical study are in such cases assumed to be separate.

The number of heterogeneous definitions is particularly great in social theories and conceptions. This seems to be primarily conditioned by the heterogeneity and plurality of layers of the domains under consideration, and their complex hierarchical structure.

There is no doubt that a description of the structure of state institutions in an antagonistic class society (to wit, special groups of armed men, prisons, etc.) could not result in a scientific conception of state in an antagonistic society. To reveal the essence of the state, it is necessary to point out the causes which gave rise to the state, its goals in an antagonistic society, etc. To achieve this aim, one has to go down to the basic layers of social being, those that are more fundamental from the point of view of causal explanation, that is, one has to pinpoint the economic causes that produced the state and account for its existence, its main political purposes, etc. Heterogeneous definitions in the social sciences serve not only to distinguish the object defined from other objects but also to reveal its essence. It is not necessary therefore that each of the distinctive features in the Dfn of definitions in social sciences should change the extensional of the concept defined. Considering the purpose of the definition, these features are not redundant (although that is precisely what they are within the formal approach). That is why we frequently come across definitions that are more compact on some occasions and more extended on others, although their definienses are extensionally equal.

The most concise definition of state in an antagonistic society was given by Lenin in his lecture 'The State': 'The state is a machine for maintaining the rule of one class

over another.' [1] A more extended definition of this type of state may be given: the state is a political organisation of the economically dominant class serving as the instrument for the exploitation of the oppressed classes.

It is easy to see that the definienses of both definitions are extensionally equal.

In defining imperialism, Lenin lists five features characterising its political and economic essence. He also writes there: 'If it were necessary to give the briefest possible definition of imperialism we should have to say that imperialism is the monopoly stage of capitalism. Such a definition would include what is most important....'

And he adds: 'But very brief definitions, although convenient, for they sum up the main points, are nevertheless inadequate, since we have to deduce from them some especially important features of the phenomenon that has to be defined.' [2]

As we have seen, in defining antagonistic classes Lenin points to the features distinguishing them from each other, and then gives a short definition: 'Classes are groups of people one of which can appropriate the labour of another owing to the different places they occupy in a definite system of social economy.' [3]

The definitions cited here have been borrowed from the theory of historical materialism. Since historical materialism is a synthetic science generalising the data of other social sciences and substantiating the materialist view of society, languages of various levels of cognition of social life are used in defining concepts here. We establish the nature of phenomena defined at higher levels of the hierarchy of social structure in terms of phenomena of the lower and more fundamental levels. For example, in defining antagonistic classes Lenin proceeds from the analysis of the relevant economic relations, stressing the particular significance of the connection between separation of society into classes and the corresponding production relations. He defines state in an antagonistic society by bringing out its political essence as an instrument of exploitation (the function of suppressing the dominated classes). In defining

[1] V. I. Lenin, 'The State'. *Collected Works*, Vol. 29, p. 478.

[2] V. I. Lenin. 'Imperialism, the Highest Stage of Capitalism'. *Collected Works*, Vol. 22, p. 266.

[3] V. I. Lenin. 'The State'. *Collected Works*, Vol. 29, p. 421.

imperialism as a historical epoch, Lenin again turns to the economic and political essence of imperialism. We have a similar situation in defining socio-economic formations.

In many social sciences heterogeneous definitions are the rule rather than exception. In any case, the languages of phenomenological description, explanation, and evaluation are all used in these definitions.

One and the same object (natural or man-made) may be used in different functions in human activity. Difference in functions is sometimes registered in different names for the same object. Thus, a piece of cloth may be called a rag, if we use it for sweeping crumbs off a table; it will be called a table-cloth, if we use it for covering the table, or a curtain, if it is used for draping over a window. Depending on the object's function, its different properties are shifted into the foreground as the essential ones.

In social activity, groups and classes attempt to use certain objects in their own interests, laying emphasis on those aspects or properties of the objects, which satisfy their interests. We therefore often come across different definitions of one and the same object: the Dfn's may be extensionally equal but the formal principle of extensionality does not work here (just as it is invalid in nonextensional contexts which do not permit substitution of one expression for another, extensionally equal to it).

Assume that we have to form a concept (and then formulate its definition) with regard to social phenomena in the synchronic plane. Our first task would then be to single out in these phenomena their common specific properties which, moreover, are essential for them; in other words, we would have to perform the identification abstraction on them. Not infrequently we have to deal with a situation where these properties are manifested *in varying degrees* in the phenomena under study. They are manifested in a greater measure in some representatives of the set of phenomena under consideration than in others. Naturally, it is easier for the researcher to determine the specific essential properties of the phenomena under analysis in those representatives where these properties are expressed more explicitly, in a *more developed*, *classical form*, and then to see how they are realised or manifested in the other representatives.

The classics of Marxism-Leninism used this method ex-

tensively. Thus, in elaborating the concept and the corresponding definition of the slave-owning socio-economic formation, they used as their model Greece and Rome at a certain period when the slave-owning system reached its most mature, classical form. Definitions of other antagonistic socio-economic formations were also constructed by them on the basis of analysing models satisfying the requirements indicated above. Lenin analysed imperialism and elaborated the corresponding definition by considering certain classical models—the highly-developed capitalist countries. It is in this connection that Marx remarked that the key to the anatomy of an ape lies in human anatomy.

Application of definitions elaborated by this procedure raises some difficulties precisely because they cannot be applied in a purely formal way to the representatives of the set that do not belong to the classical models, all the more so when these representatives are considered under different historical conditions. A given theory (with the corresponding definitions) should be applied in a concrete and meaningful way, not abstractly and formally, proceeding from the dialectical correlation of the general and the particular.

Definitions sometimes assume the nature of ideal norms if they relate to the sphere of creative activity, behaviour, and education, are based on an analysis of the objective laws of the development of society and take into account the goals and tasks being solved by the society, and sometimes assume the nature of ideal norms if they rely on the study and generalisation of models and are oriented at the future. Of this kind are definitions of some ethical, pedagogical, and esthetic concepts. They do not merely generalise what there is in the reality, drawing on its best models: they also comprise a certain regulatory element (that which one should strive for) setting the direction of men's activity and behaviour.

CHAPTER 6

METHODS OF FORMULATING DEFINITIONS AND THE DIFFICULTIES ARISING IN THE PROCESS

6.1. METHODS OF FORMULATING ANALYTIC DEFINITIONS AT THE LEVEL OF NATURAL LANGUAGE

Methods of formulating analytic definitions have been elaborated by the Polish logician T. Kotarbinski. He distinguishes four methods of constructing and substantiating analytic definitions: inductive, word-building, philological, and intuitive.

The *inductive method* of constructing analytic definitions at the natural language level is the principal one. It was extensively used by the philosophers of antiquity in elaborating definitions of ethical concepts (e.g., Socrates, Plato, and Aristotle). Kotarbinski reproduces Aristotle's analysis of the concept of 'magnanimity'. To work out a definition of this concept, Aristotle considers some individuals of whom it may be said that they are magnanimous. The difficulty of singling out the common specific features of the objects analysed in this case (that is, the difficulty of applying the identification abstraction to them) lies in that the term being analysed is not always used in the same sense in different contexts. Thus, Aristotle shows that men like Socrates are called magnanimous because they maintain mental equilibrium under all thinkable circumstances. Men like Ajax and Alcibiades are called magnanimous because they left no insult go unpunished. Kotarbinski observes, however, that apart from these meanings the term 'magnanimity' has another (and primary) meaning: it is applied to persons who 'selflessly render generous assistance to their wards regardless of their own losses and labour, caring naught for the danger and the expense, bearing no animus against those who do harm to them, and displaying all possible tolerance and sense of dignity...'.[1]

[1] T. Kotarbinski. *Op. cit.*, p. 631.

Revealing identity and differences in the meanings of the sign expression being analysed calls for establishing *synonymity* (identity) of the sign expressions. Two sign expressions are regarded as *extensionally synonymous* (having the same meaning) if they have the same signified (the denotatum). If that is not the case, they are believed to be extensionally different, not identical. These principles are applied at a meaningful level in formulating analytical definitions.

The *homonymity* of the terms under analysis is fairly simply revealed on the basis of these principles when a term denotes sets of objects that have no common elements (for instance, the term 'country' is used to denote states; it is also used to denote rural areas; spheres of interest and affiliation; a jury, etc.). The boundaries between the various meanings of the word 'country' are marked rather clearly.

In defining the concept of magnanimous man in the example considered above we ran into a more complicated case. The fact is that a person like Socrates may simultaneously be magnanimous in more senses than one: he may keep his mental equilibrium under all circumstances and possess the qualities which Ajax and Alcibiades possessed, being at the same time magnanimous in the third, common sense. The sum of magnanimous persons in the three senses indicated here and their complements has a special name with a clearly delineated meaning, viz. the name 'man'. This name may be regarded as the name of the corresponding object domain.

When we come across the word 'country', the denotata are so different from one another that it is possible to state unambiguously in which of the meanings the word is used in the given context. The uncertainty of contextual differences is here compensated for by the definite differences in the denotata.

In the case of contextual definition of the meaning of the term 'magnanimous' the situation is made even more complicated by the fact that the predicate 'to be magnanimous' is dispositional. The property 'to be magnanimous' (in each of the three meanings indicated above) is manifested only in the relevant situations and in such a way that in some situations a person may show magnanimity in all the three senses. This makes the analysed contexts extremely vague. For the definition of the meaning of the

word 'magnanimous' to be unambiguous, an analysis of contexts of great length and a comparison of a number of contexts are required. The dispositional nature of the properties under analysis makes the object domain extremely indefinite, too, if it comprises not only persons but also their moral qualities.

It is apparently impossible to determine all the shades of meanings of the word 'magnanimous' with absolute precision. The fact is that different contexts are of different value for analysis, as they are produced by persons of different educational background and linguistic ability. Besides, these contexts are practically infinite in number. That is why analysis of natural language alone produces only approximate results. The linguist introduces a synthetic element in definitions constructed in this way: he introduces the necessary precision and rigour, standardising the lexical meanings. He thus resorts to the 'simplifications' and 'deadening' that are a constant feature of the process of cognition, as V. I. Lenin pointed out.

The *word-building method* of formulating analytic definitions consists, according to Kotarbinski, in explaining the meanings of sign expressions through their etymological analysis. Thus, we may define democracy as the power of the people. This definition results from the translation from the Greek of the two component parts of the word 'democracy' (δήμος—the people and κράτος—power). Etymological analysis leads us in this case to the definition of one of the meanings of the word 'democracy'. According to the *Philosophical Encyclopedia*, the term democracy is used: 1) to denote the power of the people; 2) to characterise a state which is marked by certain legal features (recognition of the will of the majority as the source of power and declaration of the citizens' freedom and equal rights); 3) as a synonym for citizens' rights and freedoms.[1]

The etymological meaning of the word 'democracy' is the definition of only one of the meanings of this word at the present level of knowledge. Etymological analysis in this case leads us to a definition of one of the meanings of the word.

When etymological analysis does not result in a modern

[1] *Filosofskaya entsiklopedia*, Vol. 1, p. 456.

definition of even one of the meanings of a sign expression, it cannot be viewed as a means for formulating definitions. Thus, etymological analysis of the word 'tragedy' would not yield a definition of the meaning of this word in its present-day interpretation. The word consists of the words τράγος (goat) and ώδά (singing). Etymological analysis would result in an absurd definition of tragedy as the singing of goats. This analysis, however, is significant from the point of view of the history of culture, for it reveals the genetic roots of this kind of drama (resp. of theatrical performances). The fact is that tragedy as a kind of theatrical performance emerged from rounddance rites whose participants dressed as goatlike satyrs.

Kotarbinski distinguishes the so-called *philological method* from the above methods of formulating analytic definitions. As distinct from the inductive method, no use is made here of reference to and analysis of extralinguistic examples, analysis being limited to contextual definition of unfamiliar words of a foreign language.

Since we formulated the general definition of definition in such a way that it covers establishment or specification of terms within the framework of the given language *S* or its extensions, we (as distinct from Kotarbinski) do not consider the procedures of contextual definition of unfamiliar words of a foreign language and their translation into one's native language to be definitions of these unfamiliar words. However, if the contextually established meaning of a native or foreign word is included in the native vocabulary this mode of establishing meanings should be regarded as a definition.

By the *intuitive method* of elaborating an analytic definition Kotarbinski means a certain *Gedankenexperiment* involving analysis of the denotatum of a certain term. He writes in this connection: 'Thus, many ask: "What is a nation?", and try to answer it as follows. They imagine a certain concrete nation, for instance, the French; having previously outlined mentally the set of features characterising the French as a nation (such as common decent, common history, common state, a common language of their own, a common spiritual culture of their own), they start thinking: "What would happen if the French lost their state, as the Poles once did: would that mean the end of their existence as a nation? And suppose they assimilated a foreign tongue,

having forgotten their own?" Finally, these and similar mental operations result in a definition of nation free from inessential features.' [1]

6.2. METHODS OF FORMULATING CREATIVE SYNTHETIC DEFINITIONS

When we take a purely formal approach to analysing definitions and consider them within the framework of a closed system, all synthetic explicit definitions may be viewed as abbreviative definitions. The method of formulating them consists precisely in introducing a new sign expression in the system as an abbreviation of another (more awkward) expression which is constructed according to some rules in the development of the system.

In 3.5 we analysed the creative character of definitions in connection with extending the deductive possibilities of a closed theory. We mostly considered those synthetic definitions that are not creative. However, some synthetic definitions are creative. This is the case, first, when their introduction into a theory results in its transformation, so that propositions that could not be proved or explained in the original theory become provable and explicable; when some of the propositions of the original theory prove to be false in the light of the definition introduced; when the basis of the theory is changed, etc. Second, definitions are creative when they are used to *construct* new objects of a theory. They *change the nature of the researcher's work*. Their introduction may result in essential extensions of the theory.

H. Weyl stressed the creative nature of synthetic definitions in the second sense, in regard to meaningful mathematical disciplines. By creative definitions he meant those which record effective methods of constructing new objects, particularly those objects that he called 'ideal elements'. He contrasted these definitions with those used to introduce the so-called derivative relations through applying certain formal combinatorial procedures.[2] The class of definitions through formal combinatorial procedures, accord-

[1] T. Kotarbinski. *Op. cit.*, pp. 633-34.

[2] H. Weyl. *Philosophie der Mathematik und Naturwissenschaft*. Leibniz Verlag, München, 1928, pp. 3-4.

ing to him, includes definitions of those objects which are defined through explicit descriptions of their properties and relations between them, rather than through operations with them. 'In this way we subordinate genetic construction to the being-at-rest of the relations; however, later we shall, on the contrary, substitute constructive processes for all the relations.[1] A noncreative definition of this kind may be illustrated by the definition of parallelism ($g \| g'$):

$$g \| g' \equiv \neg \exists x \{(xg) \wedge (xg')\}$$

(the straight lines g and g' are parallel if and only if there is no such point x that would lie both on the straight line g and the straight line g'; the expressions xg and xg' mean respectively: 'the point x lies on the straight line g', 'the point x lies on the straight line g'').

The distinctive feature of these definitions is that, within the framework of abstract meaningful mathematical theories, they are based on the assumption that all objects (not merely elementary ones) exist before they are introduced by definitions. Still, ideal objects are constructed by creative definitions.

Explaining the way in which ideal elements like infinitely remote points are introduced into Euclidean geometry by definitions, Weyl writes: '"Infinitely remote points" of Euclidean geometry, in which parallel straight lines allegedly intersect, are just such ideal elements, linked to the real points by a creative mathematical definition. More generally, one can proceed in this way from the geometrical images of a limited but accessible part of the space R to the introduction of the inaccessible points as ideal ones (inclusive of the infinitely remote), thus extending the limited space to the complete space of projective geometry. It is necessary to establish here, through geometric construction in R, when several real, that is, passing through R, straight lines radiate from one ideal point. This point is easiest defined as the apex of some three-sided angle (formed by real straight lines). Thus one comes to the explanation: "Three straight lines a, b, c which do not lie in the same plane but each pair of which lies in the same plane, define a certain 'ideal point' (a, b, c)".'[2]

[1] H. Weyl. *Op. cit.*, p. 5.

[2] *Ibid.*, p. 8.

The original operational domain is also extended and generalised in this way, through addition of ideal elements. A classical example of this is the introduction of imaginary numbers into mathematics.

These extensions frequently involve meaningful interpretation of expressions which emerge in the development of scientific theories, and have to be viewed as meaningless in the light of current definitions.

Thus, the original definition of the operation of subtraction included a restriction: '$a-b=c \equiv b+c=a$, where $a \geqslant b$.' When $a=b$, the expression '$a-b$' was interpreted as absence of a number; zero as a number in its own right was introduced later. If $b>a$, the expression '$a-b$' was believed to have no meaning. To give this expression a meaningful interpretation, negative numbers had to be introduced into mathematics. The definition of subtraction could then be formulated without this restriction. The formulation of the new definition thus entailed introduction of new objects in the domain, extension of the subtraction operation to the domain, and lifting of the restrictions indicated above. This means that the generalised definition of subtraction was synthetic in nature, although the term 'subtraction' had been previously used in science.

Application of definitions through abstraction (see 1.8) in various sciences always entails formulation of definitions of a synthetic nature: these definitions are used to reveal in the objects under study certain properties, not amenable to direct perception, which mark corresponding sets. New names are introduced for these properties and the relevant sets.

The synthetic, creative nature of many definitions (in the first sense) is frequently revealed when the theory of experiential knowledge is not divorced from the empirical level of knowledge but, on the contrary, constantly assumes the analysis of facts and phenomena that precedes both the formation and improvement of relevant theories and conditions, the transition from one theory to another. Definition of the concepts of a theory is in this case an integral part of their formation, of generalisation of experimental measuring activity and the construction of relevant hypotheses. The synthetic, creative nature (in any sense) of definitions introduced into a theory is in this case beyond doubt.

6.3. THE PROCESS OF CONSTRUCTIVISING REALITY AND THE DIFFICULTIES OF FORMULATING DEFINITIONS

Many difficulties of formulating definitions are due to the so-called process of constructivising reality. We deal with this process at all stages and levels of cognitive activity.

What does this process consist in? Material reality is essentially *dialectical*. This means that every object is continually changing in some of its characteristics (and does not remain the same, in this sense); there are no hard-and-fast lines between separate individual objects. If that is so, the question arises, for instance, what is the basis for applying one and the same proper name or definite description to an individual object at different times (e.g., application of the same name to the same person in different periods in his life, application of descriptions to individual trees, brooks, etc., in different seasons, etc.). Such applications of words and phrases may only be justified if we are able to identify *separate objects* as such in time and distinguish them from one another in space. *This process of identification and distinction is called the process of constructivisation of reality in the narrow sense.*

Logically and genetically, the constructivisation process in the narrow sense is the precondition for further development of cognition and adequate use of language. Given the continually extending sphere of our experience, constructivisation of real objects in the narrow sense is the necessary condition of the progress of knowledge at all of its levels and stages. The progress of cognition involves the elaboration of new complex methods of constructivising reality that sometimes contradict common sense (such are some methods of identification in classical statistical physics and in quantum statistical physics).

At the initial stages of human knowledge the constructivisation process in the narrow sense was directly and essentially linked with social practice.

The objective basis for constructivising reality is the dialectic of motion and rest. Identification of objects as such is based precisely on the singling out of the relatively constant and invariant in the changing objects, the rela-

tively identical or invariant being raised to an absolute in the process of cognition.

The presence of a 'resting' invariant in objects does not determine a rigidly unambiguous division of reality into discrete units by the subject. This is convincingly proved by the data of ethnography and linguistics. Thus, some peoples that stand at a relatively low level of social development introduce divisions of reality that are absent in others. In Tasmanian, for example, there are special names for river in general, for a big river, for a very big river, and for a very small river. The Eskimos have some twenty words to denote ice in the various stages of its formation and melting.

The division of fluid reality, whose parts are connected by continuous transitions, is determined not only by the relative identity and distinguishability inherent in the objects themselves but also by the needs of social practice. Many kinds of rivers are distinguished and given special names in Tasmanian and many kinds of ice are distinguished and named in Eskimoan for the reason that rivers play an important role in the lives of Tasmanians and ice, in the lives of Eskimos. The existence of special names in these languages for different kinds of the objects in question ensures a greater reliability in men's communication and mutual understanding in production. Lenin pointed out that practice forms part of the definition of the object in the sense that it serves as the basis for delineating the boundaries of objects and thereby their specificity [1] (for instance, the way in which the boundary between the vegetable and animal worlds is drawn determines the specific features of these domains). Practice forms part of definition also in the sense that the modes of using one and the same object are sometimes regarded as specifying features of the objects themselves (and the corresponding concepts) so that the same object is given different names. The idea that in reflecting reality we resort to constructivisation in the narrow sense involving transformation of the continuous into the discrete, division of whole into parts, transformation of the relative into the absolute,

[1] See V. I. Lenin. 'Once Again on the Trade Unions, the Current Situation and the Mistakes of Trotsky and Bukharin'. *Collected Works*, Vol. 32, Progress Publishers, Moscow, 1975, p. 94.

etc., is contained in Lenin's well-known thesis concerning the dialectical nature of reflecting motion. He wrote: 'We cannot imagine, express, measure, depict movement, without interrupting continuity, without simplifying, coarsening, dismembering, strangling that which is living. The representation of movement by means of thought always makes coarse, kills,—and not only by means of thought, but also by sense-perception, and not only of movement, but every concept.

'And in that lies the *essence* of dialectics.

'And precisely *this essence* is expressed by the formula: the unity, identity of opposites.' [1]

Already at the level of constructivising reality, the process of reflection is creative and dialectically contradictory. This is manifested, for instance, in the fact that qualities corresponding to the categories are cognised through their opposites: the continuous through the discrete, changes through their invariants, wholes through their parts, the relative through the absolute, etc. These dialectical difficulties of reflecting motion were discussed already in antiquity and are still being discussed in modern science.[2]

Constructivisation of reality in the narrow sense is the result of complex contradictory reflection of reality and, in its turn, *a condition for the subsequent deeper reflection of reality*. Further cognition is hardly imaginable unless the objects of cognition are identified as such and distinguished from one another, that is, unless they are constructivised (in the narrow sense). If the constructivisation has been carried out, we may, for instance, identify various objects with one another, form the relevant sets and thus distinguish them from one another, drawing hard-and-fast lines between them.

Constructivisation in the narrow sense together with establishing hard-and-fast lines between sets of constructivised objects will be called *constructivisation in general*. This process may serve as the basis for further simplification of reality, for classification and establishing systematic links and relations between sets of objects.

Formulating definitions of individual objects at any

[1] V. I. Lenin. 'Conspectus of Hegel's Book *Lectures on the History of Philosophy*'. *Collected Works*, Vol. 38, Progress Publishers, Moscow, 1961, pp. 259-60.

[2] See S. A. Yanovskaya, *Op. cit.*, pp. 232-34.

level of knowledge involves constructivisation of reality in the narrow sense. Definition consists in explicit formulation of specifying characteristics and procedures for singling out individuals.

Definitions of the same objects may intentionally differ from one another: they may be specified and singled out in different ways. When the separate objects of the domain of reality under consideration have been constructivised, we may form classes of objects by applying, e.g., the identification abstraction and describe them on the basis of definitions. Thus, having studied and defined individual chemical elements, we may define the classes of metals and non-metals, the correctness of defining metals and non-metals depending on the level of knowledge of a sufficiently large class of individual chemical elements. Each element gets a proper name precisely for the reason that it plays a very significant role in man's theoretical and practical activity, but sometimes such proper names for each object are not introduced.

Constructivisation of each individual, the setting up of sufficiently precise and hard-and-fast boundaries between individuals and classes of lesser generality does not entail, however, the setting up of just as rigid boundaries between classes of greater generality. We may thus draw sufficiently hard-and-fast lines between the species of mammals, fishes, birds, etc., but drawing a constructivising boundary between the animal and the vegetable worlds involves considerable difficulties.

It is very difficult to draw constructivising boundaries (and elaborate the relevant strict definitions) between certain social phenomena, between the domains of different sciences. Disputes arise not only about defining philosophy or drawing a constructivising boundary between science and art, but also about defining mathematics, drawing a constructivising boundary between mathematics and logic, etc. Drawing such boundaries and elaborating corresponding definitions are facilitated when we begin to operate with abstract objects (different from first-level abstractions) and idealised objects. Thus, in mathematics we operate with natural numbers that are sets of sets equivalent to certain standard sets (see 1.8). We treat these sets as individual but abstract objects and give them proper names: 1, 2, 3, 4, ...

In physics we operate with abstract objects that are physical magnitudes (e.g., mass, time, length, speed, acceleration). Such idealised objects as 'material point', 'ideal gas', 'absolutely black body', etc., are also introduced into scientific theories. These abstract and idealised objects are constructed by applying various mental procedures to real objects. In forming the concept of natural numbers, an important role is played by definitions through abstraction (see 1.8), and in forming the idealised objects indicated above, by a special type of *Gedankenexperiment* called idealisation (see 6.4).

When these objects are introduced into a theory as primitive ones, their definitions, as a rule, are not given at the level of the theory in question: they are substantiated at the level of methodological discussion. Other abstract objects of the theory are explicitly defined within the framework of a closed theory through elementary objects (including objects that are operations).

When the objects of research are very complex, science sometimes suggests various ways of constructivising them. The question of selecting the best constructivisation and rejecting all others is solved in an indirect way by taking into account their fruitfulness in science and in resolving practical problems.

Let us explain this by citing some concrete examples.

Almost all sciences suggest extensionally different specifications of their subject matter. Thus logic is defined in a narrow and broad sense, and each of the definitions is supported by rather weighty arguments. There are also distinctions between the various definitions of the general concept of man's diseases. In defining instruments of labour, all specialists include among these the mechanical instruments of labour (implements of production). However, opinions differ as to the inclusion of the following components among labour implements: (1) 'the vascular system of production' (pipelines, various containers); (2) production shops, railways and highways, waterways in operation, canals, means of communications, etc.; (3) land as production area and as the source of properties affecting the object of labour; (4) the natural forces harnessed by man (steam, electricity, the energy of light, wind, rivers, etc.).

A similar situation arises in defining material production, which most authors describe as production of means of

subsistence or production of material wealth. Thus, the *Philosophical Encyclopedia* defines it in the following way: 'Production is the process by which man transforms natural objects for satisfying his needs, in which his activity mediates, regulates and controls exchange of matter between himself and nature.'[1] Engels interpreted production in a broader sense: 'According to the materialistic conception, the determining factor in history is, in the last resort, the production and reproduction of immediate life. But this itself is of a twofold character. On the one hand, the production of the means of subsistence, of food, clothing and shelter and the tools requisite therefore; on the other, the production of human beings themselves, the propagation of the species. The social institutions under which men of a definite historical epoch and of a definite country live are conditioned by both kinds of production: by the stage of development of labour, on the one hand, and of the family, on the other.'[2]

In mathematics, constructivisation of objects by definitions also frequently gives rise to heated controversy. Suffice it to recall here the debates between representatives of the classical and constructivist trends in mathematics and logic and the various definitions of infinity (actual and potential, considered within the framework of the potential realisation abstraction), negation (classical and constructivist), etc.

In these examples of extensionally unequal definitions (where the Dfd is, however, represented by the same term), direct experience cannot be the umpire deciding the choice of one definition over another. A definition is chosen by applying the criterion of *productivity*. Definitions are believed to be more productive if *scientific theories and conceptions constructed on their basis allow the solution of a greater number of scientific problems* (and that in a more effective, constructive, and rigorous manner), *an effective use of the results obtained in practice* (in the widest sense of this term); *if the theories and conceptions, constructed on the basis of accepted definitions, possess a greater heuristic potential*. Since science always chooses the more productive

[1] *Filosofskaya entsiklopediya*, Vol. 4, p. 387.

[2] F. Engels. 'The Origin of the Family, Private Property and the State.' In: K. Marx and F. Engels. *Selected Works* in three volumes. Volume Three, Progress Publishers, Moscow, 1976, p. 191.

and easily applicable means, everything that is less productive and fruitful is usually declared to be unproductive and unfruitful.

Assume that we have two extensionally unequal definitions where the extents of the definiendum and definiens of the first definition form normal parts of the extents of the definiendum and definiens of the second definition, with the Dfd's being the same.

Does this mean that if one of these definitions is true, the other is in all cases necessarily false? No, it does not. The controversy concerning preference of one definition over another will involve establishing their productiveness or unproductiveness, their great promise or lack of it, their relevance or irrelevance, usefulness or harmfulness.

The dispute as to which of the definitions should be accepted and which discarded frequently ends in both definitions, both modes of constructivising reality being accepted as fruitful and expedient, but fruitful and expedient in different relations and for attaining different scientific and practical goals. This means that two concepts are simultaneously introduced into science.

They are not always labelled by different terms, but instead a distinction is made between using the term *in a broader and narrower sense*, specifications being sometimes introduced in the form of various adjectives (absolute definitions become relative, see 1.4).

Thus, different concepts of infinity coexist at present in the classical and constructivist trends of mathematics serving the goals of its theoretical constructions developed within various approaches and conceptual schemata. Infinity, which is differently defined in constructivist and classical mathematics, is called by different phrases, each of them comprising the term 'infinity' (e.g., 'actual infinity', 'Cantor's infinity', 'realisable infinity', 'infinity on the basis of the potential realisation abstraction'), which marks the genetic and logical bond between the given concepts of infinity.

We believe that extensionally broader or narrower definitions of concepts considered in science do not exclude each other but may coexist.

The definition of implements of labour is a case in point. It defines a certain system that includes a number of interconnected components rather than independent elements

united by some common properties only: a description of the common components of implements of labour is the basis for identifying the most important conditions of material production in different countries and at different historical epochs. This system may be described more comprehensively or it may be represented as a homomorphic simplification of the original one. We believe that such homomorphic simplifications are quite justifiable by the goals of the theory being constructed.

The theories of social development based on the more comprehensive definitions and those that are their homomorphic simplifications will differ essentially as to the range of social phenomena falling within their scope.

As noted above, there are two interpretations of material production by the classics of Marxism-Leninism. Sometimes they restricted material production to the production of material wealth, and at other times, they described material production as the production of the society's immediate life. Marx stressed on more than one occasion that production is also direct consumption which is in its turn production, and that production cannot exist without this second form of it. 'In the first, the producer objectifies himself, and in the second, the thing he has produced personifies itself.'[1]

In evaluating Engels's book *The Origin of the Family, Private Property and the State*, where the broad view of material production as production of a twofold nature is taken, Lenin wrote: 'This is one of the fundamental works of modern socialism, every sentence of which can be accepted with confidence, in the assurance that it has not been said at random, but is based on immense historical and political material.'[2]

In analysing the laws of the capitalist mode of production, and in developing the theory of proletarian socialist revolution and proletarian dictatorship, the classics of Marxism-Leninism found it important to single out that part in the production of immediate material life which covers production of material wealth, production of means of subsistence (that is, to construct its homomorphic sim-

[1] Karl Marx. *Grundrisse der Kritik der politischen Ökonomie (Rohentwurf) 1857-1858*. Anhang 1850-1859, Dietz Verlag, Berlin, 1953, p. 12.

[2] V. I. Lenin. 'The State'. *Collected Works*, Vol. 29, p. 473.

plification). It was important, since socialist revolutions abolish private ownership of the means and implements of production of material wealth.

However, when the task of constructing communist society arose, it was quite natural to turn to the broader interpretation of material production, namely, to the interpretation of material production as production of a twofold nature. This will make it possible to analyse a broader scope of economic production relations from scientific, materialist positions (in particular, the family as an economic cell of the society, relations between families in the process of social production, etc.). This can serve as the basis for a deeper analysis of other relations as well (e.g., the relation between the individual and the social), for bringing out the laws of the development of the family, for determining trends in the development of forms of the social division of labour, for solving the problems of education and upbringing of the rising generation at a higher theoretical level.

6.4. EXAMPLE AND DEFINITION

In 1.11 we considered ostensive definitions that are an important means of the formation of the primary lexical meanings of words in teaching a natural language. In mastering the meanings of words ostensively, we merely show the object and name it without resorting to other forms of verbal activity. In scientific activity we use *analogues of ostensive definitions* that will be called *exemplary definitions*, or definitions through citing examples.[1]

Let us discuss briefly the role of exemplary definitions in science.

We know that Euclidean geometry was constructed as a concrete meaningful theory with a certain previously chosen interpretation serving to describe the properties of the physical world. In one type of approach this interpretation may appear as something secondary with regard to the abstract meaningful axioms describing a certain abstract set of systems of objects. However, in constructing a model

[1] P. Lorenzen. *Logik und Grammatik*. Dudenverlag, Mannheim, 1965, S. 8.

(including the preferred one describing the properties of the physical space) we explain the objects of the model by citing examples.

Thus, we have no means for explaining the incidence relation (the term 'to lie in' and the corresponding proposition 'the point P lies in the plane E') other than direct demonstration. An explanation of this kind naturally involves a lower level of abstraction in the study of objects: relatively smooth surfaces are substituted for planes, and areas for points; in other words, idealised or properly geometrical objects are replaced by their imperfect realisations. That means that we resort here to nonrigorous exemplary definitions. The latter are necessary for the fruitful development of science, as they show what material objects are transformed into ideal ones, connect the abstract-formal and the concrete-meaningful, and facilitate the growth of the heuristic potential of science. At the same time exemplary definitions, in a sense, do not violate the necessary rigorousness of the axiomatic theory, as the relevant axioms fix only such aspects and properties of the primary objects as are necessary for the development of the theory.

Since we are dealing with interpreted Euclidean geometry describing the physical properties of space, all explicit definitions of the more complicated objects formulated on the basis of the primary ones will prove to be no stricter than the exemplary definition of incidence.

Proceeding from the primary objects, we may use explicit definitions to introduce more complex objects and construct them with the aid of various instruments (e.g., a pair of compasses and a ruler). The figures constructed according to the definition will always differ from the objects described by definitions. Thus any triangle we construct will have a certain area, length of sides, angles, etc., while its definition does not mention these characteristics. This redundant information, however, is useful in that it permits to use drawings to prove theorems and at the same time does not interfere with formulation and proofs of theorems in general form, as in formulating and proving theorems about triangles in general we do not use any information in the drawings other than that which is contained in the relevant definitions (in particular, in the general definition of triangle). This enables us to use a concrete individual

drawing in proving a theorem, this being done once for all the infinite variety of triangles.

We use in this case the rule of Locke (the rule of introducing the universal quantifier):

$$\frac{P(a)}{\forall x\, P(x)}$$

(if some property belongs to an arbitrary but fixed element of a set, it also belongs to all the elements of this set).

We know that the primitive objects of physical theories, (e.g., various physical magnitudes) in closed theories, apart from having constitutive definitions (e.g., definitions through systems of equations), may also be defined operationally (see 4.3). Outside the framework of theories the primitive objects of theories and the methods of constructing them are also given definitions (be they nonrigorous). They are sometimes introduced by means of an example. The same may be said about methods for introducing idealised objects into a theory. For instance, Einstein and Infeld explained the method for introducing such an idealised object as inertia into mechanics and simultaneously defined the concept of inertia as an idealised object at the level of scientific methodology in the following manner. Assume that we are pushing a cart along a path.[1] The cart moves for some time after the push and then stops. There are various ways of increasing the path traversed by the cart after the push: e.g., lubricating the wheels, smoothing out the path. The easier the motion of the wheels and the smoother the path, the farther will the cart go. Lubricating the wheels and smoothing out the path reduce the external influences on the moving body and decrease the friction. It may be experimentally established that the less the external influences on the moving body, the longer the path traversed by it. We can think of ever new ways of decreasing the external effects on the moving body and correspondingly ever new methods of lengthening the path traversed by it, but it is impossible to eliminate all the external influences entirely. However, the law that we have established (the regular dependence of the length of path traversed by the body on external influences) enables us to conclude that

[1] See A. Einstein and L. Infeld, *The Evolution of Physics*. Simon and Schuster, New York, 1942, pp. 36-38.

if we eliminate the external influences on the moving body entirely, it will move infinitely (and uniformly along a straight line) or be at rest. This conclusion was once made by Galileo.

This example is characteristic in that it naturally allows *abstractions* and *generalisations* transforming it into a *definition* of inertia through a description of a certain *Gedankenexperiment* on the behaviour of real moving bodies. This transformation of an example into a definition involves, first of all, a transition from the consideration of a concrete moving body, a cart, to the consideration of moving bodies in general.

At the same time it is an example of the type of *Gedankenexperiment* that is called idealisation. Applying appropriate abstractions and generalisations to this example, we can obtain a definition of idealisation (or one type of it, at any rate).

Idealisation is a *Gedankenexperiment* that includes the following elements:

(1) by changing certain conditions of the existence of the object under study, we decrease their effect (resp. increase it);

(2) it becomes apparent in the process that some properties of the object uniformly change, too;

(3) assuming that the effect of the conditions on the object under study has been reduced to zero or has reached some invariant, we make a mental transition to an *extreme* case and thereby to some idealised object.

The realisation of abstractions and generalisations by which one proceeds from an example to the corresponding definition is conditioned on consideration of other examples (in this case, of the modes of introducing into science of such idealised objects as 'absolutely solid body', 'ideal gas', 'absolutely black body', etc.). Their analysis is of great heuristic significance: they indicate what abstractions and generalisations the researcher has to perform. These examples are frequently used not only for didactic purposes but also at those levels of cognition where the definition of the required scope of generality has not yet been elaborated.

Some objects can only be given exemplary definitions. Thus, inherence of a certain predicator in an individual object may only be demonstrated by concrete examples

in separate elementary sentences of a natural language. 'If we restrict ourselves to elementary propositions only, predicators will always be defined by examples, as I would put it.'[1] In other words, to be able to orientate ourselves in the world we live in, to store and process information about it, we should memorise some facts (e.g., the connections between certain individual objects and their properties) on the basis of experience including verbal experience (defining them through examples). These connections may then be substantiated by logic, as the same individual propositions may be deduced from other propositions, e.g., on the basis of the rules of Aristotelian syllogistics.

By applying rules of logic, we may obtain from general propositions of the type 'he who is reasonable is just', 'he who is just, is not kindhearted' (1) the corresponding elementary propositions which establish either the inherence of the predicator 'just' in some individuals or the noninherence of the predicator 'kindhearted'. Thus, if it is known that 'John is just', we can obtain from this proposition and proposition (1) the proposition 'John is not kindhearted'. In cases like this, as Lorenzen puts it, the connection between the predicator and the object is determined terminologically *(terminologisch bestimmt)*.

Exemplary definitions, just as ostensive definitions, are not definitions in the proper (logico-semantic) sense, but they may be regarded as definitions in the broader methodological sense (see 2.2). As is known, ostensive definitions are used to introduce into the learner's language the meanings of terms unfamiliar to him (for this reason they are synthetic for him, see 1.3). For the teacher they are analytic, as they are somehow defined in his language.

'Examples or exemplary definitions are used, first, to explain, for didactic purposes, the meanings already known to mankind; they are in this case analytic in the language of a science or scientific theory but play the role of synthetic definitions in the learner's language. Second, exemplary definitions are used to extend the language of science as they introduce new terms into it. Definition is in this case synthetic in nature. Its introduction is linked with singling out new spheres and fields of reality and with their constructivisation. Third, exemplary definitions are also used

[1] P. Lorenzen. *Op. cit.*, p. 8.

in specifying already existing terms and their definitions. Synthetic definitions are employed in this case, which specify the constructivising boundaries between objects. Finally, in explicit formulation of the meaning of an already existing term given contextually, we consider various instances of its application and create the conditions for its adequate definition. In this case we deal with an analytic exemplary definition.

6.5. EXTENSIONAL EQUALITY OF DEFINITIONS AND ITS LIMITATIONS

It has been pointed out already that reality may be constructivised in different ways. First, the same object may be singled out by describing its different characteristics. In this case, we deal with extensionally equal but intensionally different definitions. Second, there are situations in cognition when the same term is introduced to designate at least two different classes one of which is a normal part of another. This gives rise to extensionally unequal definitions.

The extensional approaches to concepts and definitions of concepts are sometimes taken in logic and mathematics (both at the meaningful and formal levels of their construction and analysis) and in some theories of mathematical natural science: two concepts (and their definitions) are believed to be indistinguishable if they have the same extent; or, two terms are identified in regard to their sense if they have identical meanings. These propositions are usually referred to as the principle of extensionality. This principle is somewhat differently formulated with respect to definitions. Let us assume that the Dfd's in definitions D_1 and D_2 are the same. It may then be said that the two definitions, D_1 and D_2, are extensionally equal if their Dfn's have the same extent, that is, if they correspond to the same sets of objects. Taking the syntactical approach, one may say that two definitions, D_1 and D_2, are extensionally equal, if Dfn_1 and Dfn_2 of these definitions are mutually interchangeable in any standard contexts.

If the definition D_1 in some meaningful logico-mathematical theory is replaced by the extensionally equal definition D_2, the set of theorems deducible in theory T_1

by the application of D_1 (when D_1 is added to the axioms and earlier introduced definitions and theorems of the theory T_1) will coincide with the set of theorems deducible in the theory T_2 by the application of D_2. Thus, the de fintions 'A square is an equiangular rhombus' (1) and 'A square is an equilateral and equiangular parallelogram' (2) are extensionally equal, since the Dfn's in (1) and (2) have the same extent. Within the framework of some theory T they may replace each other: the sets of theorems deduced by applying these definitions will coincide with each other.

If a more formal approach is taken, it may be said that definitions (1) and (2) are extensionally equal since their Dfn's, when reduced to the primitive terms of the theory, will be represented by identical expressions, and the Dfd's will prove to be eliminable (see 3.5).

Assume that we have two axiomatic theories, T_1 and T_2, with equal sets of primitive terms (or objects). Let the systems of axioms (the axiomatic definitions of primary objects) be different, but it will be assumed that the same stock of logical means will be used in deducing theorems from axioms. Then the systems of axioms (axiomatic definitions) will be *extensionally equal* for T_1 and T_2 if every theorem in T_1 is a theorem in T_2 and vice versa. Extensional equivalence of the definitions may also be established in the following way: the definitions D_1 and D_2 are extensionally equivalent in some theory T if (and only if) by adding D_1 to T we shall obtain D_2 as a theorem and vice versa, by adding D_2 we shall obtain D_1 as a theorem.

At present there are many definitions of the concept of algorithm that are based on specifying the term 'algorithm' used in mathematics before these specifications were introduced. Each of the authors of these specifications tries to elaborate an analytic definition of this concept. As S. A. Yanovskaya points out, a great number of such specifications have been introduced since 1930s. These include 'the Turing machine', 'Church's λ-definability', 'Post's finite-state combinatorial processes', 'the Gödel-Kleene recursive functions', 'Markov's normal algorithms', 'the Kolmogorov-Uspensky algorithms', 'the Yershov operator algorithms', 'algorithms defined by the Kaluzhnin graph-schemes', and many others. All these definitions are known to be equivalent, each of them entailing others.

Since all of them satisfy the requirements of mathematical rigorousness, their equivalence may be strictly proved.[1]

The following question arises in this connection: why do intensionally different definitions of the same thing appear in mathematics and logic? Disregarding the psychological aspects of scientific creativity and the didactic considerations in teaching, we may single out two groups of causes.

1. The development of scientific knowledge calls for continual specification and the improvement of earlier introduced concepts and definitions of the same objects in an effort to achieve greater strictness, the intensionals of the expressions corresponding to the Dfn's not infrequently being changed. Increased strictness of definitions in its turn provides greater possibilities for more rigorous proofs of theorems and theoretical conceptions of greater heuristic potential. 'Suffice it to note that the most significant results in the field of mathematical analysis in the 19th century were obtained after (and as the result of) a number of discussions in which the basic concepts of analysis were specified: the concepts of the real and complex number, limit, continuity, function. As the result of the discussion of the function concept, trigonometric series came to play the role of one of the most important means of expressing functional dependence in mathematics and mathematical physics. Cauchy's "fundamental series" first broke the vicious circle, in the domain of real numbers, of defining "limit" by reference to "real number" and "real number" by reference to "limit", thus making it possible to define precisely the basic concepts of mathematical analysis—"continuity", "derivative", "integral", and others—and to prove theorems about them in a rigorous way; moreover, they subsequently broached a new set of problems of distinguishing "uniform" and "nonuniform" convergence and continuity and later the more general problems of modern topology and functional analysis.'[2]

2. The introduction of intensionally different but extensionally identical definitions is sometimes determined by the goals and tasks that have to be solved by a certain theory or its fragment, by the level of analysis. In plane geometry,

[1] See S. A. Yanovskaya. *Op. cit.*, p. 248.
[2] *Ibid.*, pp. 246-47.

a circle is usually defined as follows: 'A circle is a closed flat curve in which all points are equally removed from one point that is called the centre and lies in the same plane as the curve.' Also in plane geometry, in solving a special class of tasks, the circle is defined in the following way: 'A circle is a locus subtending the given segment (the diameter) at right angles'. One of the definitions may be used more effectively for the solution of some tasks and the other, for solving other tasks.

In another theory, the definition of the same object may be formulated in terms which partly do not occur in the first theory. Thus, from the point of view of analytical geometry, a circle is a second-order curve for which the equation in the rectangular coordinate system xOy has the form $(x-a)^2+(y-b)^2=R^2$, where R is the radius and a and b, the coordinates of its centre. This definition, as distinct from those above, offers the possibility of studying the circle with the instruments of algebra and analysis based on the method of coordinates.

It frequently occurs that one type of definition is formulated at one level of analysis and for the solution of some tasks, while another type of definition (equivalent to the former extensionally) is formulated at a different level of analysis and for the solution of other tasks. This may be illustrated by the definitions of force cited above (see 4.4). Therefore, despite the extensional equality of D_1 and D_2, we cannot regard them as interchangeable at certain levels of analysis and given certain research goals.

In the theories of mathematical natural science, the primitive objects of a theory are usually accepted without definition (e.g., in classical mechanics the primitive undefined terms are length, time, and mass). These objects are introduced without definitions at the conceptual level, but their introduction is followed by descriptions of experimental measuring procedures specifying them, that is, by operational definitions (cf., for instance, Newton's definition of mass which appears in the argument preceding the construction of the theory, and the operational definitions of mass included in the theory). Conceptual definitions are used to reveal the specific features of that which is to be studied by strict physical methods, while operational definitions ensure effective definitions of concrete quantitative physical magnitudes.

Axiomatic constructions of meaningful mathematical theories and, consequently, axiomatic intratheoretical definitions are frequently preceded by explicit definitions of those objects that form a system described by the axioms (e.g., explicit definitions of point, straight line, plane, etc., in Euclid). The goal of these definitions is to ensure a certain intuitive clarity as to the nature of the objects appearing in the axioms, to find out what they are when they are used in isolation from the axiomatic constructions. These definitions play an enormous heuristic role in cognition. When they are formulated outside theories, and prior to their axiomatic construction, it would be more correct to say that the objects figuring in the axioms are not defined explicitly *within the framework of a closed theory only* rather than that they are not explicitly defined at all.

In some experimental sciences (e.g., the social ones), that are by no means constructed after the model of the logico-mathematical theories or the theories of mathematical natural science, the task of proving theorems is not the primary one. Here this task is usually supplanted by the problem of constructing hypotheses that should agree with empirical data, by the need to explain facts by laws. The principle of extensionality is applied in such theories on a very limited scale. Two definitions having the same extensional but different intensionals are not identified with each other in this case, as a rule: that definition is preferred in which the object is singled out by more significant features (see 9.2).

Choosing a suitable definition from a number of extensionally equal ones is conditioned on an analysis of the features of the definition from the point of view of their significance, productiveness, and heuristic value.

6.6. DEFINITION AND THE PROBLEM OF A MORE SIGNIFICANT AND LESS SIGNIFICANT FEATURE

As we know, philosophers have always discussed the essence of things, its manifestations, and its cognition through phenomena. In modern philosophy and methodology of science, this problem is sometimes transformed into a discussion of the forms of scientific explanation.

We shall consider the problem of the essence of things only from the point of view of distinguishing between the more significant and the less significant features used in definitions as specifications of the objects being defined; we shall not concern ourselves with the ontological aspect, but only with the epistemological and methodological ones, taking into account the various spheres, levels, and goals of cognition.

The question of what properties to choose for specifying the object being defined may be formulated primarily in regard to analytic nominal definitions, to real definitions corresponding to analytic ones, and also in regard to synthetic definitions appearing as the result of essential specifications of analytic definitions. The thing is that when the problem of choosing a name for a newly discovered phenomenon or object arises, that is, when a semantic synthetic definition is constructed (see 1.2 and 1.3), the phenomena or objects discovered are characterised, as a rule, by a single set of features, which are comprised in the Dfn; the problem of choosing a set of features does not arise in this case.

The problem of a significant feature arises when the Dfd is viewed as an object existing independently of any definition, when it has a special name in some language, and when we have considerable information as to its properties. The problem of the more significant and the less significant features is only formulated in regard to real definitions and analytic nominal definitions.

The distinction between the more significant and the less significant features becomes meaningless for some spheres of knowledge, if it is only a question of the deductive potential of the relevant theories. This is the situation in logico-mathematical theories and the theories of mathematical natural science.

We have already pointed out in 2.1 that any extensionally equal but intensionally different definitions of the same object (e.g., in Euclidean geometry) are indistinguishable as far as their deductive possibilities are concerned. Assume that we have given a twofold definition of rhombus: 'A rhombus is a quadrangle in which two diagonals are mutually perpendicular and divided into two equal parts at the point of intersection' (1) and 'A rhombus is a quadrangle having equal sides' (2). In Euclidean geometry

they will be indistinguishable from the point of view of the basic tasks solved by that science; by adding definitions (1) and (2) to the system of axioms and earlier proved theorems, we shall obtain the same corollaries (theorems). Definitions at this level and in this sphere of scientific knowledge are believed to be equipollent if they are *extensionally equal*. If we take the view that the objects of Euclidean geometry in some sense pre-exist their definitions, it may be said that the properties recorded in the Dfn's of the definitions (1) and (2) are extensionally equal and the definitions (1) and (2) themselves are equivalent in the sense indicated above. Consequently, the intensional differences between the Dfn's of the definitions (1) and (2) are in this respect irrelevant. That means that the problem of distinguishing between the more essential and the less essential features (properties, in this case) does not arise here.

We have a similar situation in the sufficiently mature theories of mathematical natural science (e.g., in classical mechanics), if abstract objects are involved having the nature of physical magnitudes for which variables are introduced in the relevant mathematical formalism. However, here again different definitions of the physical magnitudes may be given at the qualitative-descriptive, metatheoretical or interpretative levels. These definitions may prove to be nonequipollent even if their Dfn's are extensionally equal (see the examples of definitions of force in 4.4). Preferences in choosing these definitions are usually the subject of debate in natural science, stemming from the differences in evaluating the intensional characteristics of the Dfn's of definitions. However, the transition to the level of operational definitions in Bridgman's special sense (if we take into account the principles of reducibility formulated in the above) or to the level of constitutive definitions (see 4.3 and 4.4) makes it possible to identify all the extensionally equal definitions of the same object.

This means that the assessment of intensional characteristics as less significant or more significant becomes meaningless in the spheres of knowledge indicated here and with regard to the objects mentioned (with respect to establishing the deductive possibilities of a theory). Preferences for some definitions over others are not determined here by the Dfn recording the essential or inessential (the more essential

or the less essential) but by other parameters: simplicity, naturalness, fixed traditions, etc.

Why is no distinction drawn here between the more significant and the less significant? The reason is, we believe, that in mathematics we deal with abstract and idealised objects in forming which we have already made an abstraction from everything that is inessential, secondary, uninteresting to mathematics and have singled out those characteristics that are of prime importance for mathematical activity.

We know that 'pure mathematics deals with the space forms and quantity relations of the real world—that is, with material which is very real indeed. The fact that this material appears in an extremely abstract form can only superficially conceal its origin from the external world. But in order to make it possible to investigate these forms and relations in their pure state it is necessary to separate them entirely from their content, to put the content aside as irrelevant'.[1] It will be of some interest to note that Aristotle pointed to the extremely abstract nature of quantitative relations. '"Quantity" [he wrote] means that which is divisible into two or more constituent parts of which each is by nature a "one" or a "this". A quantity is a plurality if it is numerable, a magnitude if it is measurable. "Plurality" means that which is divisible potentially into noncontinuous parts, "magnitude" that which is divisible into continuous parts.'[2]

Specific for mathematics is precisely the study of quantitative relations. In various collections or sets (e.g., in a collective of human beings), mathematics singles out through abstraction objects which are a 'one' or a 'this', that is, which are homogeneous, abstracted from their qualitative features but at the same time distinguishable from each other, which makes them countable. In this way we obtain certain abstract sets whose properties are the domain of mathematics.

Defining the sets, relations among them, and establishing operations performed on them enables one to construct

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[1] F. Engels. *Anti-Dühring*. Progress Publishers, Moscow, 1977, pp. 53-54.

[2] *The Works of Aristotle*. Vol. VIII, Oxford at the Clarendon Press, 1908, 'Metaphysica', Book V, Chapter XIII.

descriptions for a great number of different real situations considered at the level of isomorphism.

Various geometrical figures are also constructed on the basis of abstraction applied to real objects included in social practice.

Sets and figures as objects of mathematical study appear in an abstract idealised form, only those features being abstracted from the real objects that are *essential* for mathematical treatment. That is why the distinction between *the essential* and *the inessential* becomes meaningless in mathematics at the theoretical level. This *distinction*, however, *retains its significance at the metatheoretical level*, when we are trying, e.g., to substantiate outside the limits of the theory our understanding of the primitive concepts indefinable within the framework of the theory (as done by Euclid, for instance).

We have a similar picture in physical theories. We single out here certain characteristics of material bodies by abstraction and transform them into abstract bodies of a special kind which assume the nature of physical magnitudes. At the level of object language, intensionally different operational definitions of such concepts as mass, force, length, temperature, etc., are usually identified on the basis of the principles of reducibility indicated above (see 4.2). However, the concept of the essential and the nonessential, of the more essential and the less essential is of considerable importance at the level of physical theories, too, when the problem arises of defining such model objects as the atom or the elementary particle, or when we are dealing with conceptual (not operational) definitions of a great number of notions at the methodological philosophical level, the level of explanation and interpretation.

Thus, the distinction between the more essential and the less essential is valid within those spheres of knowledge and in solving those problems where the principle of extensionality does not work.

The notions of the essential and the inessential, of the more essential and the less essential continually recur in the methodology of the qualitative-descriptive study of reality and the interpretative level of theoretical knowledge. We encounter these modes of studying reality in one form or another at various levels and stages of empirical sciences. The qualitative-descriptive methods in em-

pirical sciences have special significance at the initial stages and levels, where the need arises to identify and distinguish the objects under study by their rigid characteristics. To compile the relevant classifications and nomenclatures it is desirable to identify and distinguish objects according to criteria that are at the same time essential.

For instance, Aristotle paid much attention to specifying genera and species of animals in connection with their classificatory descriptions clearing up many of the obstacles emerging here. He noted that it was difficult to specify animal genera and species by a single feature, that a distinctive feature for a genus or species must have this property that its absence in other genera and species should make it possible to distinguish the given genus or species unambiguously. 'Now if man was nothing more than a cleftfooted animal,' he wrote, 'this single differentia would duly represent his essence. But seeing that this is not the case, more differentia than this one will necessarily be required to define him; and these cannot come under one division.'[1]

According to Aristotle, the essence of the object is manifested in the definition, in which the genus indicates the essence of the Dfd shared by it with other objects while the *differentia specifica* indicates the specific essence, that is, the essence inherent only in the given Dfd. However, interpreting definition as a proposition which permits pure conversion, Aristotle insists that one should distinguish between definitions through *essence*, through *revealing the essence of being*, and definitions *through the proper feature*. This reasoning implicitly contains the thesis that the definition (we are here dealing with real definitions) may include a proper feature of the definiendum as its Dfn, that is, a feature that is not essential—which is undesirable. The distinction between the essential and the inessential in Aristotle himself is vague enough and based on ontological reasons.

Numerous criteria for distinguishing between the essential and the inessential have been suggested in philosophy (including Marxist-Leninist philosophy). We shall set forth just a few arguments on this score.

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[1] *The Works of Aristotle*. Vol. V. 'De Partibus Animalium', Book I. 3.

First of all, the question of selecting one out of several extensionally equal Dfn's in defining the Dfd (i.e., that of determining the more essential Dfn) should be formulated concretely, not abstractly. That means that the researcher should take into account his general goal, the task being solved, the sum total of the knowledge he possesses. the system of science in which the research is being done, etc.

All other conditions being equal, out of two extensionally equal but intensionally different Dfn_1 and Dfn_2 of some definition *one should choose* that which (a) offers more possibilities for transition to the quantitative analysis of those objects which are analysed at the qualitative-descriptive level in the relevant fragment of knowledge; (b) being added to the sum of knowledge contained in the relevant fragment of science, offers greater possibilities for deducing more corollaries about the Dfd, for formulating a greater number of propositions expressing regular relations.

Let us consider an example from the history of science. We know that Lavoisier interpreted chemical element as that chemical substance which increases (or perhaps retains unchanged) its weight in all chemical reactions. Proceeding from the atomistic theory of the structure of matter and his discovery of the law of constant and multiple proportions, Dalton came to regard the atomic weight of a chemical element as its specific feature. Chemical element began to be interpreted not only as a chemical body which retains or increases its weight in all reactions; it was stated at the same time that all elements consist of identical unchangeable and indivisible atoms having the same atomic weight, and that this weight distinguishes a particular element from all other elements. Although the Dfn's in both definitions are extensionally equal, Dalton's definithan had a great advantage over that of Lavoisier.

The Lavoisier definition of chemical element contained a hint at the existence of a connection between the qualitative characteristic of an element (its indivisibility) and the quantitative one (ability to retain or increase its weight with regard to oxygen during chemical reactions). This connection, however, was established by Dalton on the basis of atomistic concepts; after his discovery, the atomic weight of elements may be regarded as the expression of those quantitative relations in which chemical elements are combined with one another.

Dalton's conception of the atomistic nature of substance, his laws, and the definition of chemical element elaborated later, served as the basis for many new discoveries, for explanations of many chemical phenomena that could not be satisfactorily explained earlier, and for formulation of a number of important laws. Suffice it to mention here Avogadro's method for determining the relative atomic weights, his formulation of the law according to which identical volumes contain identical numbers of molecules, not atoms, etc. Within the framework of this approach Mendeleyev discovered his famous periodic law of chemical elements, which in its turn entailed a change in the concept of chemical element registered in the appropriate definition. All of this means that the definition of chemical element suggested by Dalton records more important features of the definiendum than the Lavoisier definition, although they are extensionally equal.

The distinction between the significant and the insignificant, between the more significant and the less significant is of great importance in social sciences, of which we said above (see 5.2). Thus the Dfn's of definitions reflect the specific features of the Dfd's, the extensionally equal definitions corresponding to the Dfn's playing different roles in cognition at the various levels and stages of the empirical sciences. These differences between extensionally equal definitions are frequently spoken of as differences of reflecting the more significant and the less significant in definitions.

The above mentioned limitations on the distinction between the more significant and the less significant have been formulated with regard to definitions only. These distinctions are important in other respects at most diverse levels of cognition and activity (e.g., at the heuristic level). Thus the solution of any new tasks arising in science or practice always requires that the more essential should be distinguished from the less essential, the primary from the secondary. Some criteria for such distinctions are formulated by G. Polya as applied to the solution of mathematical problems.[1]

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[1] See G. Polya. *Mathematical Discovery*. Vol. I, Wiley, New York-London, 1962, pp. 287-93.

CHAPTER 7

ON THE APPLICABILITY OF TRUTH-VALUE ASSESSMENTS (TRUTH EVALUATIONS) TO DEFINITIONS

7.1. APPROACHES TO TRUTH-VALUE ASSESSMENTS IN THE HISTORY OF PHILOSOPHY

It was Aristotle who posed the question whether evaluation of definitions from the point of view of their truth or falsehood [1] was justifiable. In Book II of the *Second Analytic* he goes thoroughly into this question in connection with elucidating the relation between definitions and proofs. Aristotle's view of this problem is as follows.

The knowledge of what a given thing is so long as it exists is given by definition in the following cases. First, when we are dealing with the foundations of science which include explicit definitions of certain primitive objects and the corresponding terms; second, when the cause of the emergence of some thing is elucidated by syllogistic reasoning and, third, when proofs are used to substantiate a statement of what a given thing is.

Definitions of primitive objects do not merely register the fact that they exist or specify them: they also reveal their *essence*. These definitions are not proved *within the framework of any discipline* and in this sense *are not proved truths*. If we tried to prove these definitions we would lapse into *regressus in infinitum*. 'Either the basic premises will be demonstrable,' writes Aristotle, 'and will depend on prior premises, and the regress will be endless; or the primary truths will be indemonstrable definitions.'[2] Outside the limits of special theories (in philosophy), however, these definitions are usually given some substantiation.

Definitions of the second and third kinds are proved truths: they are true in that sense in which proved theses

[1] This chapter will primarily consider explicit meaningful definitions. Special reservations will be made when we deal with definitions in formal systems.

[2] *The Works of Aristotle*. Vol. I. 'Analytica Posteriora'. Book II, 3.

are true. Thus, according to Aristotle, we can transform an explanation of the causes of thunder which figures in a proof into a corresponding definition of thunder. Aristotle writes: 'Thus the former [definition—*Ed.*] signifies without proving, but the latter will clearly be a quasi-demonstration of essential nature, differing from demonstration in the arrangement of its terms. For there is a difference between stating why it thunders, and stating what is the essential nature of thunder; since the first statement will be "because fire is quenched in the clouds", while statement of what the nature of thunder is will be "The noise of fire being quenched in the clouds". Thus the same statement takes a different form: in one form it is continuous demonstration, in the other, definition.'[1]

The question of truth-value in definitions posed by Aristotle was discussed throughout the history of logic and philosophy (explicit meaningful definitions were meant). Alternative solutions of this problem were often suggested. Thus, J. S. Mill was inclined to believe that definitions were neither true nor false, if one singled out in the propositions described as definitions that which referred to the procedure of definition (namely, the method of establishing the meaning of a certain name) and at the same time disregarded the question of the existence or nonexistence of the objects denoted by the name defined.[2]

Mill's viewpoint and those similar to it could only be accepted in science if we did not merely disregard the question of the existence or nonexistence of the objects defined but *limited* the class of meaningful definitions to those synthetic nominal definitions which establish the synonymity of the Dfn and Dfd terms.

Suppose that a certain substance is *for the first time* discovered under laboratory conditions describable in the following way: 'the secretion of the Langerghans islands of the pancreas'. We may abbreviate this description by introducing a new term, 'insulin'. We shall then obtain the definition: '"Insulin" is a term synonymous with the term "the secretion of the Langerghans islands of the pancreas".' This definition is nominal (as the meaning of the

[1] *The Works of Aristotle. Op. cit.*, Book II. 10.

[2] See J. S. Mill. *A System of Logic. Ratiocinative and Inductive* in two volumes. Parker. London. 1951. Book I. Chapter VIII.

term is defined by establishing its synonymity with another sign expression), synthetic (as the term 'insulin' is introduced into the language of science for the first time), and syntactical (as the structure of the definition directly ensures the exchangeability of the terms Dfd and Dfn in any standard contexts). This definition contains the abstraction allowed by Mill. If all the definitions figuring in science were of this sort, they could be declared to be neither true nor false.

In many situations of this kind, however, the synthetic definition cited above would be formulated in a semantic form: 'Insulin is a secretion of the Langerghans islands of the pancreas.' We can no longer disengage ourselves from the existence of the object defined: the structure of the definition itself is designed for introducing a term for an object of some domain (in this case for a domain of material objects).

If two real definitions were offered *at the present level of the development of science* (the term 'insulin' having been already introduced into its language) namely: 'Insulin is a secretion of the Langerghans islands of the pancreas'. 'Insulin is a secretion of the thyroid gland', the former definition would apparently have to be declared true and the latter, false.

That means that the problem of applicability of truth-value assessments to definitions cannot be given a blanket solution: it should be taken into account whether the definitions are analytic or synthetic, nominal or real, semantic or syntactical.

7.2. THE VARIOUS TYPES OF DEFINITION AND THE PROBLEM OF APPLICABILITY OF TRUTH-VALUE ASSESSMENTS

Analytic nominal definitions and the corresponding real ones (that is, appearing as the result of translating nominal into real definitions) admit truth-value assessments.

In the case of real definitions corresponding to nominal *analytic* ones, it is assumed that the Dfd pre-exists the definition and is somehow distinguished from other objects by certain features that are specific to it. Therefore, such

real definitions *implicite* contain the thesis that distinguishing the object before definition is in precise agreement with its extensional distinction realised by the definition (cf., for instance, the thesis that only those animals which were regarded as lions before definition, should also be regarded as lions after definition). If this thesis is satisfied, the real definition is true, if it is not, the definition is false.

Nominal semantic (analytic) definitions (like '...is called a planet', '...is called a triangle') contain the thesis that only those objects that were called by the Dfd term before definition will also be called by that term according to the elaborated definition. If this thesis is satisfied, the definition will be true, if it is not, the definition will be false.

Nonsemantic nominal (analytic) definitions (e.g., 'The Dfd term has the same meaning as the Dfn term') contain the thesis that the terms that were believed to be synonymous in the extensional sense before definition, remain the same after definition. If this thesis is satisfied, the definition is true, if it is not, the definition is false.

Suppose that a lexicographer introduces into a defining dictionary the meaning of a term that is already in use in speech. In doing so, he analyses various speech contexts in which this word occurs, and tries to establish its meaning. The results of such analysis are first recorded in analytic semantic definitions. Thus, the meaning of the word 'pitch' may be written down in the form of an explicit definition: 'Boiled tar is called pitch', 'By pitch we mean boiled tar'.

If we bear in mind this implicitly implied proposition included by the lexicographer in the definition (which is determined by the task itself of the study), the definition cited here will be true if the word 'pitch' had really been used in speech before definition *only* for naming each of the objects having the property of being 'boiled tar' (we ignore homonymy, since it merely complicates analysis without influencing the ultimate conclusions). Specifications of this type may subsequently take the form of real definitions (e.g., 'Pitch is boiled tar'), since the term earlier introduced into the language may then be regarded simply as a representative of or a substitute for the relevant object.

As for application of truth-value judgments to synthetic nominal definitions, two cases may present themselves

here: (1) the term does not pre-exist the definition in the language, it is introduced by the definition, (2) the term exists in the language or was introduced into it by definition, the task being to introduce an essential specification of the term, that is, one involving a change in its extensional characteristics.

In the first case it is sometimes said that the object does not pre-exist definition but is created by it. In reality (in empirical sciences, at least), epistemologically the object pre-exists definition. It is created by definition in the methodological sense only, that is, as an object of cognition; its discovery is followed by a description of its properties and by naming it (the process of constructivisation).

When we introduce a name for a newly discovered, constructed or emerging object which we somehow distinguish from other objects by its observable properties, the resultant synthetic definition taking the form of a semantic definition ('let us use such and such a name for an object characterised by the following features') *cannot be directly described as true or false*. This definition establishes the meaning of a new name through a description of the characteristics of objects to which it will be applied.

Suppose that we have discovered a new secretion of the Langerghans islands of the pancreas. The problem arises of giving it a name. Assume that the discussion of this problem ended in choosing the name 'insulin' for it, which is recorded in the definition: 'Let us apply the term "insulin" to the secretion of the Langerghans islands of the pancreas.' In our view, this synthetic nominal definition does not yet allow any truth-value judgments; it is a convention regarding selection of a certain sign expression as a scientific term.

Later on, however, due to the reification of the name, that is, due to the fact that it begins to be viewed as denoting something independently of the Dfn of the definition, as current in the language and *representing* a certain object, this definition may also be regarded as true, since it may already be considered to be analytic. In particular, it may be interpreted as a real definition, too (e.g., 'insulin is a secretion of ...').

It is known that at the level of empirical knowledge we constantly have to transform the relatively true into the true in some absolute formal sense, which is evidence of

the profoundly dialectical nature of the process of cognition. This creates great advantages for cognition if only because it allows application of formal rules and apparatus to it. Definitions are no exception in this respect.

Thus the definition 'Insulin is a secretion of the Langerghans islands of the pancreas' as an analytic one is a true real definition, for it satisfies the proposition that even before the definition was applied in some concrete situation, it was this particular secretion of the Langerghans islands that was regarded as insulin. But, generally speaking, it is not excluded that this is a secretion of some other organ of animals. A real definition of insulin through a description of the structure of its molecules will satisfy the relevant thesis in some distinctly thorough-going sense. A transition from one definition to another may in this case be characterised as a transition from knowledge of the less essential to knowledge of the more essential.

Application of synthetic definitions in case (2) (when, e.g., the extensional characteristics of the terms being specified are changed) may be regarded *mutatis mutandis* as introduction of a new term since its meaning is changed. Our analysis of applicability of truth-value judgments to synthetic definitions in case (1) may be repeated *mutatis mutandis* with regard to synthetic definitions of case (2): directly, they are not applicable here either. However, they will become applicable to the same definitions as soon as the new term with a new meaning comes to be regarded as pre-existing new definitions in the language, as soon as this term is interpreted as having a more specific meaning irrespective of the definition introducing the given specification.

We may apply truth-value judgments to nominal analytic and corresponding real definitions because, in our view, they are formulated and used with regard to that part of reality which we have already constructivised; the objects of reality have already been singled out in the process of cognition by some sufficiently reliable objective characteristics and have received appropriate names by which they may be represented.

Nominal synthetic definitions are linked directly with the processes themselves of constructivising reality and its changes in the process of cognition (particularly, scientific cognition).

It may be argued that analytic nominal definitions and the corresponding real ones are true or false with regard to the existing norms of language usage, not with regard to reality, and that, consequently, these evaluations are of verbal nature.[1]

It is here important to bear in mind the distinction between definitions at the didactic level and the level of the history of cognition. If a semantic and, simultaneously, syntactical definition of the form 'We shall apply the term *A* to the object *B*' (1) is introduced into science, then the same definition rewritten in a textbook in the form 'The term *A* is used to name the object *B*' (2) may naturally be interpreted as having the character of a verbal truth: to substantiate the truth of definition (2), it is enough to refer to proposition (1). We first agreed at the scientific level to name object *B* by term *A* and then repeated that that is indeed so. However, at the level of cognition the results of which are registered in the language of science or natural language, terms introduced by synthetic nominal definitions in constructivising reality begin to acquire new content resulting from experimental study of the sphere of reality singled out by the constructivising definition, of which this study is independent. We therefore often come across scientific contexts in which definition (2) will no longer have the nature of verbal truth: the scholar may come to the conclusion that, despite the great number of new properties discovered in the object denoted by *A*, definition (2) may be retained in science. In such situation substantiating definition (2) will require an analysis of experimental facts, not only of inscriptions.

The nonverbal nature of truth or falsehood of theses contained in definitions which admit truth-value judgments is even more clearly revealed when definitions are used to specify the meanings of words in natural language or to introduce changes in the extensionals of scientific terms.

Thus the analytic definition of man as a reasonable animal capable of articulate speech (3) is in fact a proposition about extensional identity of the Dfd and the Dfn. Correct use of the term 'man' in speech showed that men distinguished themselves from animals by some properties—not

[1] In a similar way we declare, e.g., a legal norm to be true, as soon as it is, for some rational reason or other, accepted by society and recorded in appropriate documents.

necessarily those that were listed in definition (3). These properties were associated with the term 'man'. The set of men was thus constructivised before the appearance of definition (3). The latter established extensional identity between the properties of the Dfd and Dfn, and that is an empirical fact already. Even when such a definition is formulated as an analytic and semantic one ('We shall call object B by term A'), we most commonly have this proposition in mind: 'Objects designated by term A are the same objects as those corresponding to property B'. This identity relation is an empirical fact.

Every explicit definition (irrespective of whether it is analytic or synthetic, nominal or real) *no longer admits truth-value judgments* if it is interpreted as a *rule* stipulating interchangeability of the Dfd and Dfn terms (which sometimes requires their transformation on the basis of translating one into the other).

In formal systems and formalised languages, the problem of truth-value of explicit definitions is replaced by the problem of their demonstrability and is solved in a similar way. Every explicit definition extending a certain system and regarded as the rule for substituting the Dfn for Dfd and vice versa (Dfd $\rightleftarrows$ Dfn) is not a demonstrable proposition of a system (it belongs to the rules extending the stock of earlier formulated rules of system S_0).

Assume that we extend system S_0 by an explicit definition Dfd $\rightleftarrows$ Dfn, where the Dfn term is regarded as an abbreviation for the Dfd term. This definition is synthetic: it introduces a new term which is not contained in the language of system S_0. This definition is not demonstrable in S_0, it is not regarded as belonging to that language. The definition serves as the basis for expanding the language of system S_0 into the language of system S_1. When this language has been formed, our definition already belongs to language S_1. We may regard this definition as an axiom and, consequently, as a demonstrable proposition with the equivalence or equality sign (when the system is interpreted, this proposition is true).

So far we have been considering definitions where the objects for which a new name was introduced were singled out by their intrinsic objective properties. However, there is a very broad class of nominal definitions in which the conditions of using the meaning of the term introduced

by the definition are essentially determined by certain reasonable *conventions* between men. Such are nominal definitions serving to specify the meanings of sign expressions that have no sufficiently definite meaning outside the relevant contexts. Here also belong synthetic definitions introducing various units of measurement.

Let us cite some examples.

In solving a problem we very frequently specify terms with meanings that are not definite enough. Different specifications may be suggested depending not only on the nature of the problem being solved but also on the various approaches to the solution of one and the same problem. For instance, the term 'tall man' that was current in the language before definition may be specified like this: 'By tall men we shall mean only those persons who are more than 185 *cm* tall.' But the same term may be specified in a different way as well: 'By tall men we shall mean only those persons who are more than 178 *cm* tall.'

Although these definitions carry elements of objective knowledge in the sense, for instance, that men of short stature (e.g., those like Napoleon) will not be included in the class of tall men, choosing one or the other of these definitions is based on some reasonable agreement. Definitions of this sort, in our view, do not admit *truth-value judgments* (even if they are transformed into real ones).

Another situation in which the definition cannot be assessed as true or false may be illustrated by definitions of various units of measurement which are frequently introduced into science by means of synthetic definitions (e.g., 'A metre is one tenmillionth part of a quarter of the Parisian meridian'). This definition establishes a certain universal convention as to what object should be chosen as the unit of measuring length and called metre. The agreement thus established should naturally be reasonable: the system of measures is based on units that are easily reproducible and linked with practically unchanging natural objects (these considerations motivated, in particular, the subsequent transition from the 'natural' standard of length mentioned above to the standard kept at a depository). However, one might choose a great number of standards of length (natural and the corresponding depository ones) all of which would satisfy the requirement of reasonableness. *These definitions are conventions about rea-*

sonable choice of the standard of measurement. In this sense, they are neither true nor false. They may be good (if the systems of units are convenient, reliable, etc.) or bad (if they do not satisfy the conditions of convenience, reliability, etc.). Kazimierz Ajdukiewicz called these definitions arbitrary [1]—as distinct from nominal and real. The nature of conventions with which we deal when we introduce the units of measurement is determined by the needs of social practice.

7.3. UNTRUE REAL DEFINITIONS

Three types of untrue real definitions are distinguished in the methodology of science.

(1) The development of scientific knowledge results in some changes in and specifications of the relevant concepts. These changes and specifications may involve their generalisation, limitation, splitting, and, consequently, changes in their extensionals, as well as intensionals.[2] Earlier definitions of these concepts may become either too broad (Dfd$<$Dfn) or too narrow (Dfd$>$Dfn) as the result of these processes. In such cases earlier definitions have to be replaced by new ones. These earlier definitions prove to be false as they contain a thesis that is no longer true, namely, the thesis that distinguishing the object before the definition is in agreement with distinguishing it by a given definition.

(2) A situation may arise in the development of scientific knowledge when the Dfd exists (its existence may be demonstrated empirically) but the definition asserts its equivalence to a Dfn that is empty (the latter may also be proved scientifically). An empty set becomes equivalent to a nonempty one. The statement of an empty set and a nonempty set being equivalent is a false one.

(3) It is sometimes established in the development of science that both the Dfd and the Dfn are empty. These definitions are also eliminated from science as worthless.

We deal with case (1), e.g., in observing changes in the

[1] K. Ajdukiewicz. 'Three Concepts of Definition'. *Logique et analyse*, Nos. 3-4, 1958, pp. 115-26.

[2] See D. P. Gorsky. *Problemy obshchei metodologii nauk i dialekticheskoi logiki* (Problems of General Methodology of Science and Dialectical Logic). Mysl Publishers, Moscow, 1966, pp. 217-27.

concepts of function, animal, plant, curve, machine, speed, etc., which resulted in the need for changing the relevant definitions of these concepts (in the extensional sense).

Thus, the expansion of the concept of animal made it necessary to exclude some features from the Dfn (e.g., the feature of 'ability for independent movement') in order to establish a conformity between the Dfd and the Dfn (to make them coextensive), to ensure the equivalence of the Dfd and the Dfn, and to make the false proposition about the equivalence of the Dfd and the Dfn into a true one.

Once whales were included in the class of fish (case (2)) and were correspondingly defined as fishes having such and such properties. The class of whales was not an empty class, as could be proved by observation. However, there were no fishes that would have the properties indicated in the Dfn of the definition mentioned here, and this fact was duly established by science. A nonempty class cannot be equivalent to an empty class, and that was why the falsity of the earlier definition was discovered in this situation. The false definition of whales was replaced by another, true one, ensuring equivalence of the Dfd and Dfn, their coextensiveness, and agreement between definition and empirical data.

We have a similar but more complicated situation in the development of the concept of atom. Owing to the work of Dalton, Avogadro and others, the existence of atoms was believed to have been proved in 19th-century science. However, the definition ascribed to the atom, among other properties, the feature of being 'an indivisible and immutable particle of matter'. But there was no such particle in reality, as was later discovered by science. The Dfn of the definition proved to be an empty class while the Dfd (as was established by science) was a nonempty class. The falsity of the earlier definition was thereby discovered. The former definition was replaced by a new one, which did not include the properties of indivisibility and immutability in the Dfn.

Falsity of definitions in case (2) could have been proved in a different way as well—by proceeding from the analysis of the thesis which is contained in every real definition corresponding to a nominal analytic one. As has been mentioned, this thesis consists in that distinguishing an object before the introduction of the definition under analysis

is in agreement with distinguishing it by the definition introduced. In the definitions considered here this thesis was up to a certain time satisfied and the definitions were *believed* to be true. However, it came to light in the course of the development of science and social practice that this thesis is no longer satisfied: independently from the definition, it was established in science and social experience that distinguishing the object under study in social experience and in the corresponding language is not the same as distinguishing it by the given definition. That was why the definition was declared to be untrue. It was discovered, for instance, that whales are no longer distinguished in science as they were according to the earlier definition, namely, by the feature of being 'a fish with such and such properties'.

Let us now consider case (3). As is known, phlogiston was introduced into chemistry in the 18th century as a hypothetical object ('Phlogiston is a weightless substance—the element of combustibility contained in all the substances capable of burning or transforming into scale or lime when heated'). At the same time a similar substance was introduced into physics that was called caloric ('Caloric is a specific weightless substance which forms part of any body and is the cause of body heat'). The phlogistic and caloric theories that were dominant at one time proved to be erroneous; therefore nonexistence of such objects as phlogiston and caloric was proved while the corresponding terms proved to be meaningless with regard to the object domains studied by physics and chemistry. In other words, it was demonstrated that the concepts of phlogiston and caloric were empty concepts. The falsity of these definitions may also be proved by analysing the proposition which they imply.

We know, however, that these empty concepts and the theories built on them played a positive critical and heuristic role in the development of science. Engels pointed out, for instance, that 'by the phlogistic theory, chemistry for the first time emancipated itself from alchemy'.[1] The caloric theory stimulated calorimetric research. Proceeding from this theory, Sadi Carnot formulated a number of

[1] F. Engels. *Dialectics of Nature.* Progress Publishers. Moscow, 1974, p. 23.

propositions ('the Carnot cycle') which later became the cornerstone of the first law of thermodynamics.

While theories constructed on definitions of empty or nonexistent objects are capable of giving only a fraction of the objective knowledge about the domain under study, theories constructed on too broad or too narrow definitions are often believed to be full-fledged theories. Just as any experiential theory, however, they are relative in the sense that they are historically limited.

Similar situations arise in the case of propositions appearing in theories (including those that are believed to be laws of nature). Such is for instance, the proposition 'The Sun rotates around the Earth'. Proceeding from this false premise, Ptolemy nonetheless made some important (and true) scientific predictions.

When we say that some proposition is false we mean that it does not correspond to reality or (which is the same) that the situation described in the false proposition does not exist in reality. For example, 'Moscow is situated on the Dnieper , 'All metals are hard' are false propositions: they do not correspond to reality, for (which is the same) the situations of reality which they describe are empty (nonexistent). Indeed, the first proposition states that Moscow is situated on the Dnieper, but it is easy to see that such a situation does not exist. The second proposition asserts that any object belonging to the class of metals is hard. But this situation does not exist in reality either.

It has been pointed out that definitions of the form (1) may be described as untrue because the thesis contained in them is not satisfied. We shall arrive at the same conclusion if we consider the problem of truth or falsity of definitions on a broader epistemological basis. Let us consider the problem of falsity of these kinds of definitions in greater detail. Assume we are given a real definition 'A capitalist is a person who exploits other men' (that is, the proposition 'Capitalists, and only capitalists, are persons exploiting other men'). However, this situation does not exist in reality as feudal lords and slave-owners also exploited other men.

This conclusion may also be substantiated in the following way. Let us write down our definition as a proposition of equivalence: Dfd $\sim$ Dfn is equipollent to the expression (Dfd $\rightarrow$ Dfn)$\wedge$(Dfn $\rightarrow$ Dfd) (1). Let us write

our definition in the form of (1): $\forall x$ (x is a capitalist $\rightarrow$ x is a person exploiting other men) $\wedge$ (x is a person exploiting other men $\rightarrow$ x is a capitalist) (2). Since we have established that the definition is too broad, we must state that the proposition to the right of the conjunction sign in (2) is false: it is not true that if some person exploits another man he is a capitalist (the antecedent being true, the consequent may prove to be false). And that means that conjunctive proposition (2) as a whole is *false*.

Thus, when the definition is too broad, the corresponding equivalence proposition proves to be false. This conclusion also refers to a definition that is too narrow, i.e., when Dfd$>$Dfn. Let us consider the following definition that is too narrow: 'A rectangle is a parallelogram in which all sides are equal and all angles are right angles.' As an equivalence proposition this definition may be written in the following form: 'If a geometrical figure is a rectangle, it is a parallelogram in which all sides are equal and all angles are right angles, and if a geometrical figure is a parallelogram in which all sides are equal and all angles are right angles, it is rectangle.' The first proposition of this equivalence is false while the second is true (these two propositions are joined by the sign 'and'). The equivalence proposition as a whole is therefore *false*.

Definitions of the second type, considered together with the thesis they imply through comparing them with reality (by the way, this thesis is precisely what enables us to make this comparison) compel us to make the conclusion that they are *false*. Let us present the definition 'A whale is a fish having such and such distinctive features' in the form of proposition (2): $\forall x$ (x is a whale $\rightarrow$ x is a fish having such and such distinctive features) $\wedge$ (x is a fish having such and such distinctive features $\rightarrow$ x is a whale). In the conditional proposition to the left of the conjunction sign we may assume the antecedent to be true (we shall consider all those x that are whales), then the consequent of this proposition will actually prove false for any such x, and the whole of the first conditional proposition will be false (as distinct from the first kind of definitions the function 'x is a fish having such and such distinctive features' is not satisfied for any x). In the second conditional proposition the antecedent is factually false, as there are no fishes having properties that would justify their being included

in the class of whales. Since the basis of the conditional proposition (implication) is false, the implication as a whole will be true. The equivalence proposition will be *false* as a whole (by the definition of conjunction).

Definitions of the third type prove to be true, if the apparatus of classical two-valued logic is applied to them for the purpose of analysis. Let us present the definition 'caloric is a specific weightless substance forming part of every body and being the cause of body heat' viewed as an equivalence proposition $\sim$ (Dfd Dfn), in the form of (1): $\forall x$ ((x is caloric $\rightarrow$ x is a specific weightless substance...) $\wedge$ (x is a specific weightless substance... $\rightarrow$ x is caloric)). Under the same conditions as in the analysis of the previous definitions we have to state that each of the propositions forming part of the conditional propositions is false, as there are no x that would have the properties of 'being caloric', 'being a specific weightless substance...'. Due to the fact that the antecedents and the consequents of conditional propositions are factually false, each of the conditional propositions is true (by the definition of implication), while the proposition of form (1) on the whole proves to be *true* (by the definition of conjunction).

In this case the usual apparatus of modern logic proves to be inadequate for the analysis of some scientific problems precisely due to those extreme abstractions and assumptions that are accepted in the construction of corresponding logical theories. Just as this apparatus turns out to be inadequate in the analysis of propositions expressing laws of nature, counterfactual propositions, dispositional predicates, etc., it also proves to be inadequate in the analysis of truth-value of some kinds of definition.

The differences between three types of definition that we have indicated are important for the methodology of sciences and, in particular, for the analysis of the development of scientific cognition. Taking these differences into account makes it possible to reveal the distinctions in the evolution of scientific theories, concepts and their fragments determined by the various types of false definition occurring in them.

For untrue real definitions of the first type, corresponding to analytic ones, we suggest the term '*false in the weak sense*', and for those of the second type, '*false in a strong sense*'. Definitions of the third kind we shall call semi-sensible.

CHAPTER 8

RULES FOR THE INTRODUCTION AND ELIMINATION OF SIGN EXPRESSIONS APPEARING DUE TO DEFINITIONS

8.1. RULES FOR THE INTRODUCTION AND ELIMINATION AT THE LEVEL OF NATURAL LANGUAGE

The problem of rules for introduction and elimination was first raised by S. A. Yanovskaya in connection with substantiating the justifiability of higher level abstractions in mathematics.[1] Her viewpoint on the whole is that the introduction of terms for higher level abstractions in mathematical theories is justified if simultaneously the elimination rule may also be formulated for these terms.

'For science to be able to serve the goals of man's social practice, it should contain general rules and laws revealing the rigid (invariant, at rest) *essence* of things repeated in various objects and phenomena. But, to reveal the essence of things, their rigid core, the kernel, one has to make an abstraction from the inessential details. This abstraction results in abstract concepts and objects *without* which no general laws or rules may be formulated. But, to apply this law to practice, abstract objects have to be replaced by their concrete representatives: they have to be eliminated. One cannot eat abstract "fruit"—one can only eat a concrete object falling under this general concept. Any abstract concept or object in science must therefore be linked with rules for its *introduction* and *elimination*.

'The methods for introducing and eliminating the simplest abstract concepts of the "fruit" type do not present any difficulties. The difficulties begin with higher order abstract concepts (including those that may be of different orders), such as "set", "property", "function", "functional", "operator", etc.'[2] The introduction and elimination of abstractions

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[1] See S. A. Yanovskaya. *Op. cit.*, pp. 235–42.
[2] *Ibid.*, p. 235.

thus involves a vast range of epistemological, methodological, and logical problems. We shall tackle only problems of introduction and elimination of sign expressions by definitions. Let us consider introduction and elimination of terms (and corresponding objects) by means of explicit definitions of the Dfd≡Dfn form.

As is known, mastering speech already involves resorting to ostensive definitions on a large scale (see 1.11). The *rules for the introduction* of sign expressions are here the rules of ostensive definitions used by the teacher; he demonstrates the situation, shows the object, operates with it, gives the learner instructions and uses the appropriate sign expression.

The *rules for eliminating* sign expressions are here manifested in the learner's ability to find the objects by their names and descriptions, perform certain actions, carry out instructions, find situations corresponding to the simplest contexts. The sign expressions introduced by ostensive definitions are eliminated and replaced by the corresponding objects, actions, situations (that is, by their denotata). These elimination rules contain *implicite* various modes of establishing the *meanings* of sign expressions, these modes being regarded as universal by proponents of different theories of meaning. Thus the elimination rules described above imply direct correlation between a sign and its denotatum. In some varieties of the realistic theory of meaning and empirical verifiability this mode of establishing meaning is treated as an absolute.

Elimination rules also imply certain activity with respect to the objects denoted. If this activity is successful, it may be viewed as evidence that the meaning of certain signs has been assimilated (the pragmatic theory of meaning).

The corresponding introduction and elimination rules may be formulated separately for the stage of mastering speech in which no systematic study is undertaken, and for that stage of mastering speech and language which does assume systematic study.[1] In both cases we shall be dealing with rules involving verbal definitions. At these stages new sign expressions may already be introduced on the basis of other

[1] See D. P. Gorsky, 'The Problem of Meaning (Sense) of Sign Expressions as the Problem of Their Understanding'. *Logicheskaya semantika i modalnaya logika* (Logical Semantics and Modal Logic). Nauka Publishers, Moscow, 1967.

sign expressions whose meaning has already been assimilated. In the final analysis we proceed from the meanings of sign expressions that were introduced ostensively.

Before systematic study is commenced the learner's speech frequently contains incomplete verbal explicit definitions (if all of them have been converted into real ones) that is, those that do not satisfy the rule of coextensionality of the Dfd and the Dfn. The rules for introducing sign expressions will here be represented by the rules for constructing many kinds of nominal definition, in which an object described by its properties is correlated with some name (semantic definitions, see 1.2), or a name (the Dfd term) is introduced instead of another sign combination. According to elimination rules, either the Dfd term will be replaced here by the object for which the term was introduced (in the case of semantic definitions), or the Dfd term will be replaced by the Dfn.

In studying systematically, man masters new sign systems in addition to the natural language, achieving greater precision in the use of expressions assimilated at previous stages: in a certain sense, he begins studying his native language all over again. Many names assume the meaning of terms for him. To be able to use definitions and introduction and elimination rules connected with them, the learner does not have to know the definition of synonymity of sign expressions in a general form. The latter acquires great importance, however, when we begin to construct theories of logical semantics.

Elimination rules may thus be used in education and training for elaborating criteria to determine whether or not the learner has assimilated the meaning of some sign expression.

8.2. INTRODUCTION AND ELIMINATION RULES AT THE LEVEL OF SCIENCE

At the scientific level, introduction and elimination rules are used for justifying operations on higher level abstractions and idealisations. These rules may be subdivided into two classes.

The first class is related to the solution of internal problems of science. The second class involves going beyond

the limits of a scientific discipline into the domain of other sciences or revealing the correlations of a certain scientific theory and its components with reality. These rules may be formulated separately for the logico-mathematical and natural sciences.

The following rules may be suggested for the logico-mathematical sciences:

(1) introduction of abstract and idealised objects (e.g., sets, predicates, functions) by explicit definitions (Dfd=Dfn);

(2) introduction of objects (including high-level abstractions) by the ε-operator (see 1.5);

(3) introduction of objects by implicit axiomatic definitions (see 1.4);

(4) introduction of objects by implicit recursive definitions (see 3.4);

(5) introduction of abstractions by describing the modes of their formation both within and outside a developing theory (e.g., the formation of the concepts of imaginary and natural numbers).

Some of the rules listed here introduce objects only within the limits of a theory (e.g., rules (2), (4)); others may be used at the pre-theoretic level as well. Some rules comprise, apart from introduction rules, also elimination rules (e.g., rules (1), (3), (4), (5)), others are nothing but introduction rules.

The following elimination rules may be suggested for the logico-mathematical sciences:

(1) elimination of objects introduced by the schema Dfd=Dfn by replacing the Dfd by the Dfn and in the final analysis by a combination of objects corresponding to the terms of the minimal vocabulary of the theory;

(2) elimination of objects introduced by the ε-operator on the basis of proof of their existence or on the basis of proof of Hilbert's ε-theorem (see 1.5);

(3) elimination of objects introduced by a system of axioms, by constructing a corresponding model (standard or nonstandard), which often involves a transition from consideration of objects of arbitrary nature to consideration of specified systems of objects (see 1.4);

(4) elimination of objects (functions and predicates) introduced recursively, by reducing them to their computed values (see 3.4);

(5) elimination by describing the formation of abstractions outside the given theory;

(6) elimination involving applications of the relevant theory, together with its abstractions, for the solution of scientific and practical tasks arising outside the theory;

(7) elimination through lowering the level of abstractions by introducing them in the context of variables (abstractions introduced by the functional abstraction λ-operator are thus scaled down by the application operation, see 1.6).

There are elimination rules for some contextual implicit definitions (thus, in solving equations by applying a certain algorithm, we determine the values of unknown quantities satisfying them, thereby converting implicit definitions into explicit ones, see 1.6) while no such rules exist for axiomatic definitions. Rule (3) envisages another method for eliminating implicitly defined terms; to eliminate terms, we go beyond the limits of an axiomatic theory, looking for corresponding models for it.

There is a limitation on rule (4): computable functions are not, generally speaking, completely defined and therefore eliminable in all contexts (for all values of their variables).

Rule (5) is extremely significant for substantiating a number of abstractions. The point is that many abstractions and concepts came into being at the meaningful intuitive level before the formation of the relevant theories and were successfully applied in practice. Analysis of the modes of their formation may prompt the choice of those concepts and abstractions that may also be used at the level of constructing a scientific theory.

We believe that rule (6) is a fundamental one. If a theory with its abstractions is successfully applied for the solution of the problems of a different (and less abstract) theory having closer links with reality and practice, the level of abstractions is lowered and they may be given a natural material interpretation. If a theory is directly applied to technology and production, the abstractions employed therein are eliminated in a literal sense. In the latter case such abstractions as points having no dimension are replaced by material objects having three dimensions.

An object introduced by an explicit definition into a rigorous theory constructed by the genetic or inductive method may be eliminated or reduced to a set of elementary objects which, generally speaking, are not defined within

the framework of this theory. That means that in the final analysis (if a theory is constructed rigorously) justification of objects introduced by explicit definitions is reduced to justifying the introduction of the primitive, basic objects of science. The latter is, however, impossible unless one goes beyond the boundaries of the theory. In a similar way, objects introduced by implicit axiomatic definitions cannot be eliminated within the framework of the theory itself.

Elimination rules in such cases involve transcending the bounds of an abstract axiomatic theory. The search for a model satisfying the axioms of such a theory will be the rule for eliminating the system of primitive objects introduced by the theory. If the model is borrowed from the objects of a different theory, the elimination (and at the same time justification) of abstract objects will be relative. Let us note that many abstract objects and operations on them were introduced into science at the level of syntactical operations. While abstract objects like natural numbers were correlated in practical activity with a variety of material objects of different nature (these abstract objects thereby being eliminated), other abstract objects, such as imaginary numbers, were originally introduced in a purely operational way: there was neither exemplification nor interpretation available for them. However, mathematicians were able to operate on them according to certain rules and succeeded in solving problems that had been insoluble prior to their introduction (for instance, with the introduction of the number i it became possible to expand the sum of the squares of two numbers, a and b, into factors). However, to attain a more reliable justification of the introduction of these abstract objects, mathematicians began searching for a model for them, which was ultimately found.

Thus the chief method for justifying the introduction of abstractions and idealisations into science are *elimination rules that involve going beyond the bounds of a closed theory and ultimately dealing with material reality and social practice.* Introduction and elimination rules are specified in the different varieties of the construction of the natural calculi of mathematical logic.

Similar rules may be formulated for natural sciences as well. We shall point out just one significant introduction rule. The construction of many physical theories involves the introduction of idealised objects (6.4) through a *Gedan-*

kenexperiment involving the operation of extremum transition. Various forms of generalising experiments, of interpretation rules, etc., may also be included among introduction rules in science. The chief elimination rules here will be: rules for applying theory to practice; rules which guarantee agreement between computations according to formulas with direct measurement; rules which guarantee the confirmation of theoretical corollaries in experimental measuring activity, etc.

Application of elimination rules that involve transcending the bounds of a closed theory shows that abstraction- and idealisations are the result of a complicated and contras dictory *reflection* of reality. There are sometimes only remote prototypes in reality for them. Prototypes may be found even for such an abstract and idealised object as material point. N. Ye. Zhukovsky thus justified this idealisation: 'If we take only the motion of the centre of gravity into account, we shall notice that it does not depend at all on the density of the distribution of matter or the form of the body but only on the quantity of matter in the body. The centre of gravity moves as if the entire mass of the body were concentrated in it alone; thus we see in it a realisation of material point, as it were...'[1]

Thus high-level abstractions are only justifiable when both introduction rules and elimination rules exist for them. One also has to bear in mind that high-level abstractions are justified both by the intratheoretical and extratheoretical elimination rules: we have to know that the theory containing abstractions which have to be justified is successfully applied, if not directly in practice (e.g., in technology) then at any rate in solving the problems of other theories which in the final analysis admit material interpretation of abstraction used in it.

Abstractions and idealisations cannot be eliminated in the absolute sense and in all the contexts in which they occur. If that were so, it would mean that science operates with concepts that are in principle expendable.

The above analysis makes it clear that abstractions of the type 'God' cannot be used in science (they are unjustifiable, unreasonable): only introduction rules may be for-

[1] N. Ye. Zhukovsky. *Teoreticheskaya mekhanika* (Theoretical Mechanics). Gostekhizdat Publishers, Moscow-Leningrad, 1952, p. 12.

mulated for such abstractions—e.g., by means of explicit definitions (something that theologians actually do), but no reasonable elimination rules may be formulated at the level of science, and that is especially true of rules that would eliminate them outside any 'theories' and 'conceptions'. The latter requirement is a necessary one, since the abstraction of 'God' emerged in the minds of men long before any 'conceptions' and 'theories' of God.

Modern nominalism, as is generally known, wages a war against universals. Elimination of universals from science would entail a considerable restructuring of it. The weakness of nominalism, however, lies in that it fails to construct a satisfactory and natural translation of many propositions of science into nominalistic language (e.g., propositions like 'There are more *a* objects than *b* objects': 'There are more dogs than cats') [1], as well as in the fact that nominalistic language is so cumbersome that it hinders the development of science, depriving it of powerful heuristic potential. One cannot but agree with Alonzo Church that banishment of universals from science will only result in that theory will be unbearably complex if at all possible.[2] This is borne out by some fundamental scientific and methodological arguments.

Anticipatory reflection of reality, as we know, is inherent in all living matter (anticipatory reflection is the property of cell protoplasm). That means that an organism does not only reflect and adjust to the present states of the environment but also to its future states, anticipating them. That is also made possible by the states of the environment not changing chaotically but in a certain invariant succession (e.g., successions of the seasons). This constancy of successions in the environmental changes is transformed into a chemical succession of molecular processes.

Man is endowed with the property of anticipatory reflection of reality to a greater extent than other animals. The prime means of anticipatory reflection of reality is science, scientific cognition with the store of experimental, logical, and methodological means at its disposal. Universals (ab-

[1] See L. Genkin. 'Nominalistic Analysis of Mathematical Language'. *Matematicheskaya logika i yeyo primeneniya* (Mathematical Logic and Its Applications). Mir Publishers. Moscow. 1965. pp. 216-24.

[2] See A. Church. 'Propositions and Sentences'. *The Problem of Universals*. Notre Dame. 1956. p. 9.

stract and idealised objects) play a very important role in this apparatus of anticipatory reflection. Here also belong 'ideal elements' (Hilbert's expression) which have purely operational significance. It is precisely abstraction and idealisations and application of rules of logic that enable man to 'play out' mentally, in a fairly simple and accessible form, various situations before they have been practically realised or registered graphically. Banishing universals from science would mean emasculating it, depriving it of a powerful heuristic potential.

Does all this mean that analysis of science conforming to nominalistic precepts is fruitless? No, it does not. At critical periods in the development of science, these analyses do not merely facilitate greater rigorousness of theory; more importantly, they help to reveal *the constructive, effectively verifiable* essence in the theories that are not constructive enough.

CHAPTER 9

ON THE RIGOROUSNESS OF DEFINITIONS

9.1. THE CONCEPT OF RIGOROUSNESS OF DEFINITION AT VARIOUS LEVELS OF KNOWLEDGE

Definitions have to satisfy requirements of varying degrees of rigorousness depending on the level of knowledge. Explicit definitions are fairly indefinite at the level of everyday experience, for which ample proof is furnished by the defining dictionaries. Explicit definitions of the meanings of words are here supplemented as a rule by instances of contextual use of the words and phrases defined.

This indefiniteness is first and foremost due to the fact that in describing reality at the level of everyday experience we do not resort to the simplifications and models or the limitations on the object domains that are current in science. The sphere of reality encompassed by man's everyday experience is extremely varied and indefinite although the processes of constructivisation and abstraction are widely employed at this level, too. The appropriate scientific apparatus cannot yet be directly applied to this sphere of experience and language—not in their developed form, at any rate.

The primitive elementary terms (and the corresponding objects) have not yet been singled out at this level, which hinders elaboration of rigorous definitions and inevitably leads to circular definitions (see 1.13). The difficulties of arriving at adequate definitions through properties are aggravated by the fact that when the same objects are included in different spheres of practice and activity they acquire different functions and different properties. Thus the function of table as an article of furniture may be played not only by the specially manufactured objects but also by boxes, chairs, tree stumps, etc. The description of the same objects belonging, e.g., to the sphere of the psyche, frequently varies (in the extensional sense, at that) depending

on the age and profession of the person formulating the definition and on personal and collective experience. Thus, an identical emotion may be viewed as a manifestation of love by one person and as a manifestation of cowardice and egoism, by another.

Since definitions of everyday experience are analytic, embodying as they do the meanings that have already coalesced in the language, the lexicographer faces a very difficult problem. He attempts to solve it by singling out a set of meanings for one and the same word or phrase and by resorting to contextual or exemplary definitions (see 1.5 and 6.4).

It would seem, however, that at the level of everyday experience and natural language one should not demand a greater rigorousness of definitions than that which is characteristic of definitions of words and phrases contained in various dictionaries (defining, phraseological, etc.). These definitions are quite sufficient for the primary function of language—the communicative one. They are also sufficient for describing them in terms of everyday experience shared by all persons speaking a common language.

At the level of qualitative scientific description of reality involving observation and the simplest forms of experiment and subsequent classification of the objects and phenomena being studied the researcher attempts to single out in the objects the *essential* properties as their distinctive features (provided their names have already been introduced in the relevant language); he sets limits to the field of investigation, trying to establish a certain subordination between sets of different objects and conceptions of varying degrees of generality. One of the goals of such a study is the establishment of hard-and-fast lines between the objects under analysis. However, the researcher faces enormous difficulties here as reality itself is profoundly dialectic in its nature.

The dialectic of reality may remain hidden at the level of everyday experience. However, a more detailed analysis entails the need for overcoming the difficulties of dialectic nature. Thus natural languages possess a dichotomous schema for dividing living organisms into animals and plants. Within the limited sphere of everyday experience it was not difficult indeed to distinguish such animals as cows, horses, dogs, bears, from such plants as birch-trees, pines, fir-trees, etc. The former move, produce sounds, take food,

just as man does, while the latter are fixed in one place, do not produce sounds, and do not take food.

Science accepted the dichotomous animal-plant schema of alternatives and tried to elaborate the corresponding definitions of animals and plants within this framework. Man's experience expanding in the systematic scientific study of nature revealed, however, the inadequacy of the original definitions and the considerable difficulties facing the scientist. A hard-and-fast constructivising line between the fauna and flora is not feasible precisely because there are entire classes of living beings (like Appendicularia and Pharonides) which have the properties of both animals and vegetables, so that some scholars are inclined to include them among animals while others, among plants. Quite possibly the rejection of the dichotomous animal-plant schema and its replacement by a new one (e.g., a tripartite schema in which a new world of living beings and a proper name for it are introduced) would result in a simplification of the biological taxonomy and a partial overcoming of these difficulties.

The difficulties of identifying objects with themselves and with one another, of drawing hard-and-fast constructivising lines between sets of objects by means of definitions, are revealed by science in the most diverse domains of reality, which makes it necessary to change the principles of identification, to resort to idealisations, assumptions and reservations.[1]

Generally speaking, high-level abstractions and idealisations make greater demands on rigorousness and preciseness of definitions. Although abstractions and idealisations reflect certain aspects, links, and properties of material things, mental operations on them cannot, nevertheless, be replaced by operations on the objects themselves that would verify their adequate correspondence to reality. Vague and nonrigorous definitions of abstractions and idealisations lead to their unscientific interpretations.

At the level of mathematical natural science and mathematics scholars continually operate with abstractions of a higher-than-the-first order and with idealisations that have

[1] See D. P. Gorsky. *Problemy obschei metodologii nauk i dialekticheskoi logiki* (Problems of General Methodology of Science and Dialectical Logic). Chapter I, § 4; Chapter II, § 1.

extremely remote prototypes in material reality. As we have already pointed out, these sciences lay more rigorous claims on definitions. It should be noted that the concept of rigorousness is hard to define formally. For instance, in Yanovskaya's article dealing specifically with the role of mathematical rigorousness in the creative development of mathematics, the concept of rigorousness is introduced contextually, through various examples.[1]

Definitions in this sphere of knowledge should provide adequate explications of the meanings of terms and, moreover, specify the objects introduced. Objects introduced by definitions must in this case permit the application of certain formal rules which would make it possible to arrive at an unambiguous solution of problems arising in the development of science.

That is precisely why in classical mechanics, the original conceptual definitions of notions through descriptions of their properties (which actually appear at the level of metatheory) are replaced by operational definitions in which standards of measurement are introduced for a number of basic concepts (see 4.4, 4.5).

Owing to the introduction, through operational definitions, of abstract objects which assume the nature of measurable physical magnitudes, the physicist is able to employ computations instead of measurements that are difficult or, at times, in principle impossible to realise. In performing these computations, he proceeds from information that is more readily available and from laws of nature that can be written down in the language of mathematics.

Mathematical concepts that are not precise enough in the senses required, also frequently form an obstacle for solving algorithmic problems, for determining whether or not a certain rule is applicable in all cases of a given sort. That is why the founders of mathematical analysis formulated the rule for finding the derivative of the product of two functions $f(x)$ and $g(x)$, which is not algorithmic, in the following way: check whether both functions $f(x)$ and $g(x)$ have derivatives. If they do not, the rule is inapplicable. If they do, mark these derivatives as $f'(x)$ and $g'(x)$ and write $g(x) \cdot f'(x) + f(x) \cdot g'(x)$ and replace $f(x)$, $g(x)$, $f'(x)$, $g'(x)$ by the functions they denote. The expression for the

[1] See S. A. Yanovskaya. *Op. cit.*, pp. 243-50.

derivative function of the product of $f(x) \cdot g(x)$ will then be obtained. The point here is that the operation of proceeding to the limit which plays the principal role in the operations of mathematical analysis is not in general algorithmic, whereas the founders of the differential and integral calculi tried to find an algorithm which would make it possible to perform this operation (or to make certain that that is impossible) in determining the limit of the ratio $\frac{\Delta y}{\Delta x}$, that is, the derivative of y on x.

'Each time it turned out, however, that the required algorithm could not be found, until at last mathematicians understood that to answer the question, e.g., whether any continuous function in general has a derivative, they have to know what "a continuous function", "a limit", and "a derivative" are; they have to have *precise* definitions of these concepts with which one can work according to definite rules that provide an unambiguous yes-or-no answer to questions similar to the above.'[1]

It is sometimes important for the development of science to give a precise formulation of the problem itself. 'We are all aware, of course, of the great role played by introducing a more precise formulation in the solution of the famous problems of antiquity—those of squaring the circle, the trisection of the angle, and doubling the cube. For more than two thousand years these problems have resisted all attempts at solution: it was not even clear whether one could accept as solutions the numerous methods of construction (with the aid of different instruments) of a square equivalent to a given circle, of an angle that forms a third of a given angle, or a cube whose size is twice the size of a given cube—the methods that have been suggested time and again since antiquity. When these problems were precisely formulated as problems in constructing geometric figures with the aid of compass and ruler, and when the connection between these constructions and the solution of quadratic equations was fully elucidated, only then did it become possible to give rigorous proof that the problems, if thus formulated, are in-

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[1] S. G. Ibragimov. 'Methodological Prolegomena to the Construction of a General Theory of Calculi'. *Voprosy filosofii*, 1967, No. 2, p. 105.

soluble: it is impossible to carry out the necessary construction procedures with the aid of compass and ruler.'[1]

It seems beyond doubt that such fields of knowledge as mathematical natural science and mathematics need rigorous definitions in the sense indicated above for their fruitful development. The following questions, however, arise: How is this rigorousness attained? Is it absolute? Let us attempt an answer to these questions by considering mainly rigorous logico-mathematical theories.

9.2. THE RELATIVE NATURE OF THE RIGOROUSNESS OF DEFINITIONS

Precision and rigorousness of definitions of abstract objects having the nature of physical magnitudes is necessarily conditioned on introduction into a theory of idealising assumptions, introduction of idealised and abstract objects simplifying the situation under study. Thus definition of mass as measure of inertia may be viewed as rigorous enough if we assume the existence of such an idealised object as inertia and respectively the existence in nature of inertial systems in a certain absolute sense. Since such systems exist only in theory while in nature all inertial systems are relative, mass can be measured only approximately, not absolutely.

By introducing idealising assumptions and idealised and abstract objects into various physical theories, we obtain a means of describing the laws of nature in relatively simple form in mathematical language, and of formulating precise definitions of concepts. In introducing the natural number series by the Peano axioms or in constructing it inductively, we actually introduce a certain sequence of objects isomorphic with the natural series. As for the numbers 1, 2, 3, 4, . . ., with which we operate in meaningful mathematics, they are defined on a conceptual level beyond the boundaries of formalised arithmetic. In defining them we may proceed from set-theoretical concepts as, for instance, Frege and Russell did (see 1.7). This theory, however, is in its turn built on a number of abstractions and idealisations. Therefore, other classes of numbers ultimately based on the na-

[1] S. A. Yanovskaya. *Op. cit.*, pp. 247-48.

tural number concept which are consistently introduced into meaningful mathematics by definitions, will be rigorous in the same sense in which definitions of natural numbers are. And the rigorousness of the latter is relative as it is ultimately determined by some assumptions that are not justified in a purely formal way.

In the inductive definition of natural number itself (see 3.4) we used the indefinable concept 'following x' whose meaning is explained informally, by citing examples.

Just as inferences obtained by applying deduction rules cannot be more true than the premises from which they are deduced, and just as computation results cannot be more precise than the information used in these computations and obtained by measurements, so definitions of objects cannot be more rigorous than the definitions of objects that serve as the basis for their construction and substantiation (even if the latter is done outside the theory).

As mathematical concepts are constantly revised, specified and developed, it is clear that there exist in mathematics concepts of varying degree of rigorousness, that the concept of rigorousness itself is relative and historical here.

Mathematical theories do not appear all of a sudden—they imply analysis and application of the results of the old theories. Therefore mathematical theories, just as any other kind of theory, must not be considered outside the context of their history, at any rate not in elucidating certain methodological problems. In constructing a mathematical theory, concepts are often introduced in such a way that they should take into account the entire previous development of mathematics and that the terms of the theory should cover all the typical cases of their use outside the framework of the theory. In other words, when we have to use a definition to specify a term with a meaning that has already become established in science and is already used and understood at a certain level, this definition will contain a thesis, an assertion to the effect that the meaning of the term introduced by the definition corresponds to the term's meaning formed before the definition was introduced. As we have shown, such definitions are called analytic (see 1.3).

Thus if we try to define area, this definition must cover all those objects that were described as areas prior to the definition, that is, we strive to attain extensional equality of the meanings of terms before and after definition. A math-

ematically strict definition, as has already been noted, must make it possible to prove theorems within the context of a theory about the properties of the object introduced by definition. The concept that has been specified must therefore be given a certain preference over one that is being specified (for instance, it may serve as the basis for proving theorems and making inferences that could not have been proved or deduced on the basis of the definitions being specified). That means that the specified definition (respectively the meaning of the term) is not intensionally equivalent to the concept or meaning that pre-existed definition.

Up to a certain moment science, mathematics included, is satisfied with fairly imprecise meanings of the terms used and has to manage without general definitions (although they may be rigorous enough in the particular cases). Thus at the times of Descartes a continuous curve was understood as a curve that could be drawn without taking pencil off paper. This definition of continuous curve naturally could not yield the conclusion concerning the existence of curves that have no tangents at any point (the Bolzano-Weierstrass theorem, the proof of which already required precise definitions of continuity and tangent).

Already in antiquity men learned to measure areas and volumes of figures and bodies. For instance, when the area of a certain figure was calculated, the question naturally arose as to what kind of object the computed area of the given figure constitutes. The area of the circle, for instance, was defined by the mathematicians of antiquity as a certain limit of inscribed and circumscribed polygons. This definition, however, was not sufficient for defining the areas of sectors and segments. That meant that mathematics needed a general definition of area and volume, that the comprehension of the general concepts of volume and area at the imprecise intuitive level no longer satisfied the needs of mathematics.

This circumstance was pointed out by Lebesgue in his book *On Measuring Magnitudes*. A. N. Kolmogorov in his preface to the book also points out similar difficulties in mathematics. 'The question of the area of a surface is particularly acute. In elementary geometry, apart from the areas of the cylinder and the cone for which the general problem may be circumvented by evolution onto a plane, the area of the surface of the sphere is also "computed". This computation, however, has no definite sense until the con-

cept of the area of a surface has been defined. It is far from generally known that the difficulty is not at all in formulating such a definition in a textbook but in the fact that a correct elementary-geometrical definition of the area of a surface applicable in at least the simplest cases was in general found only at the end of the 19th century and is only set forth in special memoirs. Textbooks on analysis and differential geometry *define* the square of surface as the integral

$$I = \iint \sqrt{1 + p^2 + q^2}\, dx \cdot dy.$$

'The usual "proofs" that this integral really expresses the area of a surface do not bear criticism for the simple reason that one cannot prove the equality of the integral to the area of the surface without defining beforehand what an area is. This circumstance is really scandalous as far as the commonly accepted presentation of differential geometry is concerned.'[1]

On the contrary, introducing the necessary level of precision into the content of concepts, their definition at the strictly formal level often involves a radical transformation of the existing scientific theories and their rapid development. Suffice it to remember in this connection the great role played in the development of mathematics by the work of Cauchy and other mathematicians dealing with the specification of the concepts of continuity and limit. Extremely characteristic is also the specification of the concepts of algorithm or, in a different terminology, of the concept of computable (recursive) function as developed by A. Turing, E. Post, A. Church, A. Markov, A. Kolmogorov and other mathematicians.

Up to a certain point in the development of mathematics the absence of a general concept of logical inference did not interfere with such development. In deducing theorems from the axioms of the theory, the scholars were guided by intuition, and by the rules of logic formulated for the particular cases of deducing conclusions from premises of a definite form. However, the solution of problems in the foundations of mathematics made it necessary to elaborate such a definition.

[1] A. N. Kolmogorov. Preface to A. Lebesgue's *On Measuring Magnitudes*. Uchpedgiz Publishers, Moscow, 1960, p. 12 (in Russian).

In the examples cited above of specifications, by means of analytic definitions, of the concepts of the area of a surface and of algorithm, we were dealing with cases in which the meanings of terms before and after definition were interpreted as extensionally equal, as different definitions of one and the same entity. The specifications based on such definitions are therefore founded on the thesis that the *definition that has been specified is only equivalent in a certain sense to the one being specified*, the one that is 'intuitively clear' and general, but that the *specified concept nevertheless reflects the essence of things more fully and precisely*. That means that the thesis of the equivalence of the concept that is being specified and the concept that has been specified *cannot be proved purely formally* —it is an empirical fact.

Rigorous formal proofs are here impossible because one of the parts of the equivalence being proved is not strictly defined but based on intuitive considerations. Rigorous formal proofs of this equivalence are in this case replaced by nonrigorous empirical substantiation. Thus we say that the concept being specified and the concept that has been specified are extensionally equivalent, for all the heretofore known areas of surfaces retain their identity under the more precise formal definition.

If one can demonstrate that all specifications of one and the same term by different definitions are *extensionally equal* (and this can only be proved in a purely formal way, since we are here dealing with the more precise Dfn's) one obtains a new argument for the equivalence of the more precise concepts to the original one. Thus the more precise concepts of algorithm obtained by different mathematicians proved to be equivalent to one another. We thus obtain one more argument in favour of the equivalence of the general intuitive concept of algorithm and the more precise rigorous concept.

We have thus seen that in substantiating the thesis of the equivalence of the concept-to-be-specified and of its more precise formulation, we had to resort to *nonrigorous proofs*.

We usually view as absolutely strict those synthetic definitions that are used to introduce, by convention, new names for the description of certain objects that are regarded as abbreviations of such descriptions. However, if this interpretation is given to meaningful explicit definitions

(e.g., within the framework of meaningful axiomatic theories), the above considerations about the impossibility of absolute strictness of definitions hold (if the definitions are taken within the context of the entire theory and not in isolation): at the meaningful level, the Dfn expression that is being replaced includes terms which are ultimately introduced through undefinable terms whose meanings are not defined in a strictly formal way.

However, if we deal with purely formal systems, the explicit abbreviative definitions used in them have a certain absolute status of rigorousness while we remain within the purely formal approach. Assume that in some propositional calculus we define the signs $\wedge$, $\vee$, $\rightarrow$, $\neg$ axiomatically and then introduce the sign $\sim$ by an abbreviative explicit definition. It may then be regarded as rigorous in some absolute sense. The point here is that axioms define correlations between signs regarded autonomously.

If one disregards the scientist's work that results in the construction of a certain formal system or its interpretation, this formal system will be of a purely conventional nature, just as cards or chess. However, in operating with objects of formal systems, we realise, at some level of metatheory, that we are dealing with rigid constructive objects that do not change throughout the argument, that certain signs have to be identified in it, and others distinguished. The concepts of rigid constructive objects and of the identity and difference of signs are introduced in a nonrigorous, informal way. In other words, this system will only work in the presence of formally nonrigorous definitions.

A system proves to have scientific significance if some model may be found for it on the basis of interpretation. In this case the system ceases to be purely formal and its signs cease to be used autonomously. It becomes a theory in a certain object domain. Its explicit definitions at the semantic level become, just as in other meaningful theories, rigorous in some relative sense only.

Formal rigorousness of definitions of terms, propositions, rules, etc., does not yet in itself guarantee a fruitful development of science or of obtaining genuine and reliable results. In the above-mentioned work that analyses rigorousness in the creative development of mathematics, Yanovskaya cites some examples which show that formal rigorousness alone not merely fails to facilitate the development of sci-

ence but may even channel it in the wrong direction. What science needs is the kind of rigorousness that would guarantee genuine results conforming to reality. Any specifications based on definition should therefore be introduced in such a way that the theory as a whole might be applied to reality or at any rate might be applicable to other theories which have been verified and tested and which are applied to reality—let it be through a series of mediacies. If specifications are to be made in the correct direction and ensure fruitful development of science, one has to take into account the entire course of its development and its practical applications. It would appear that consideration of this development (even before the theory has found any applications) offers the scholar certain guarantees that some specifications are useful, productive, and reasonable, while others are not. In the empirical sciences, the need for specification of concepts and attendant alterations in the theories is frequently conditioned on the original theories with their unspecified concepts becoming inadequate: they cannot be used to describe new experimental data in the language of theory.

Formal rigorousness should always be determined by the essential criteria, by its conceptual usefulness and by the authenticity of the knowledge obtained. That is what creates genuine correctness ensuring fruitful development of science.

But, if meaningful considerations play a dominant role in science, can it be that formal rigorousness may be dispensed with? No, it cannot. The fact is that many scientific problems are insoluble unless formal mathematical apparatus is applied that is based on formally rigorous definitions. Through applications of logic and mathematics, the solution of problems based on material activity, on practice (which is sometimes difficult or even in principle impossible to realise) can be replaced by the solution of problems in the sphere of intellectual theoretical activity. Formal rigorousness combined with the meaningful approach and the 'essential criteria' will result in a creative cognition of reality.

Reasoning in everyday life and in qualitative description of reality also requires a certain rigorousness in definitions, although in these cases, as we have already pointed out, rigorousness should not necessarily answer the demands

made on mathematical theories and the theories of mathematical natural science.

Supposing that one wants to buy a tablecloth; it is necessary to measure the area of the table. It is quite obvious that one does not need to measure the area of the table to within a micron. In a similar way, it does not make much sense to explain to a five-year-old what a natural number is by giving the relevant inductive definition or the Frege-Russell definition. The child will in this case be quite satisfied with an exemplary definition.

That means that the level of rigorousness is again determined by the nature and specifics of the problems being solved, that is, by certain meaningful considerations.

CONCLUSION

Definitions are a most important component of any scientific theory or conception, of a more or less complete chain of reasoning in which something is proved or disproved. A discussion of the problems of definition in conjunction with the processes of cognition, and the structures and functions of scientific theories (something that traditional logic mostly failed to do) extends and enriches the theory of definition in a great measure. Thus enriched, it becomes an important field of the logic of scientific knowledge and methodology of science.

Within the approaches indicated above, the development of the theory of definitions involves the employment of formal logical means of analysis and, moreover, application of the ideas and methods of dialectical logic, since elaboration of a definition or selection of one definition over another are always conditioned on elucidation of the complex problems of the genesis and development of knowledge.

The functions of definitions in cognition are extremely varied. It is these functions that determine the role of definitions in scientific cognition and in reasoning.

I. Definitions of the primitive concepts of a scientific theory largely determine its content. The way in which the primitive concepts of a scientific discipline are defined both through explicit and implicit (axiomatic) definitions helps to shape its content and its explicative and heuristic possibilities in a significant degree.

The classifications accepted in a scientific discipline, the mode of ascribing membership in certain classes to individual objects, the domain of application of the laws formulated in it depend on the way in which the various concepts are defined in it. Thus, our conception of thinking,

or mental activity, will be responsible for including certain highly organised animals among the thinking or nonthinking beings. Our definition of this concept will answer the question whether or not highly sophisticated machines are endowed with intellect.

II. Definitions of scientific theories formulate more or less 'rigid' rules for identifying and distinguishing the objects under study. No orientation in our environment, not even orientation of the most elementary kind, is possible without distinguishing and identifying objects. However, there is mostly no need in everyday life for *formulating* rules for identifying and distinguishing objects in the shape of more or less strict scientific definitions. We identify and distinguish objects using our sensory experience and our ability to determine the meanings of words in speech from the context.

As for scientific theories, we cannot manage here without definitions as means of distinguishing and identifying objects. A general theory cannot be constructed unless specific features of the objects are revealed in investigation.

As we know, law-governed connections are usually formulated in regard to classes of objects. To apply these laws to real objects, we must have relatively rigid criteria determining the membership of a certain object in some class. These criteria for identification and distinguishing the objects of the domain under study are recorded in the relevant definitions.

Elaborating these criteria sometimes involves great difficulties, one of the reasons for which is the absense of hard-and-fast boundaries between objects owing to the dialectical nature of material reality.

III. Definitions in axiomatic theories are an effective means of obtaining new truths. As noted above, axioms in such theories are used to define (at any rate partially) the meanings of the fundamental primitive concepts of a theory in an implicit form. These concepts are then used to define other concepts of the theory. From the axioms themselves (that serve as implicit definitions of the basic concepts), corollaries (theorems) are derived according to certain rules. Their content is determined by the system of axioms and the deduction rules used in obtaining corollaries. Although the propositions being proved are contained *implicite* in the axioms, they are brought out and formulated in an explic-

it form in the process of deduction thereby yielding new knowledge, new truths.

IV. Definitions are an important means of introducing new terms into science, they play an important role in the construction of scientific terminology in any field of knowledge.

For words and signs used in a scientific theory to assume the nature of scientific terms, they must be nonambiguous. That means that a term must denote one single object. This requirement is tantamount to the demand for exclusion of homonymy from the language of science. The meaning of the term is established by definition. An important source of scientific terminology is one's native language as well as other languages, the so-called dead languages in particular. Definitions specify the words of the native language and introduce new words that are not current in it, transforming them into terms of scientific theories.

V. Definitions play a great role as a means of abbreviating cumbersome and lengthy descriptions and complicated expressions current in scientific disciplines.

Definitions are used for abbreviating discourse in different sciences. Generally speaking, objects could be defined with the aid of complex descriptions registering only their specific properties (these descriptions figure as definienses in definitions). But this would hinder greatly both assimilation of scientific knowledge and the progress of science.

To demonstrate this, it would be sufficient to formulate any proposition of a scientific theory through description rather than in those terms of the theory that are substituted for it. It would be easy to imagine, for example, how cumbersome the proposition 'system entropy is proportionate with the logarithm of its probability' would become if we substituted the unambiguous descriptions of the relevant objects for the terms, 'entropy', 'system', 'proportionate', 'logarithm', 'probability'.

But that is not all. In describing such concepts we shall encounter new terms which will have to be replaced by new descriptions and so on until we reduce all the terms (resp. all the concepts) to certain primitive ones accepted without definitions on some rational grounds. If science did not eliminate these numerous cumbersome relations, it could hardly develop at all.

VI. In monologues and dialogues (arguments, discussions)

in which certain propositions are substantiated or refuted, we must unambiguously establish the meanings of words—at any rate those that form part of the proposition being defended or refuted. That implies that we have to use definitions to formulate explicitly the meanings of the relevant sign expressions. One should bear in mind two circumstances here. First, whether a proposition will be proved or refuted depends on the mode of definition of the sign expressions appearing in it. Second, if the opponent and the proponent adhere to different definitions of the terms of the proposition being discussed, the results of the discussion (if it can be completed at all) cannot be viewed as scientifically valid.

As the reader may have seen, this book poses numerous logical and methodological problems. Many of them have been barely touched upon in the literature. The author is therefore far from thinking that the solutions of problems suggested in the book may be regarded as comprehensively analysed and final.

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